BBM 403 Combinatorics and Graph Theory
Hacettepe University

Lecture 4:
Matchings in Graphs

Lecturer: Lale Özkahya

Resources:
Reinhard Diestel, “Graph Theory”

Outline

Matching and r-factor

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A matching that contains all vertices of a graph G is called a perfect matching (or a 1-factor) of G.

Alternating Path and Augmenting Path in Bipartite (A, B)-graph

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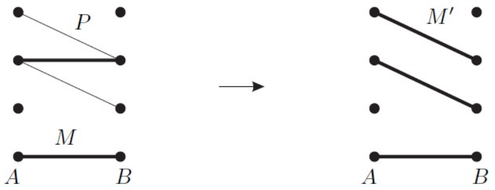

Figure: Augmenting the matching M by the alternating path P

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Next: Show that a minimum vertex cover has also at most $|M|$ vertices.

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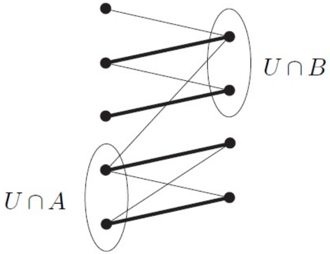

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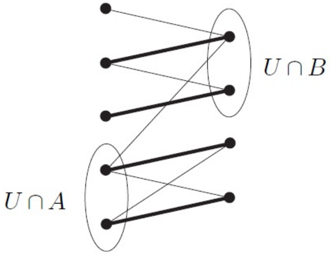

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Theorem (Hall, 1935): G contains a matching that saturates A if and only if $|N(S)| \geq |S|$ for all $S \subset A$.

First proof of Hall's Theorem

Proof by induction:

- Apply induction on $|A|$. For $|A| = 1$, clearly the theorem holds.
  Let $|A| \geq 2$ and assume that Hall's condition is sufficient of a mathing that saturates A when $|A|$ is smaller.

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  for every $S \subset A \setminus \{a\}$.
- G' contains a matching that saturates $A \setminus \{a\}$ by inductive hypothesis, this matching together with ab is a matching of G.

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- $G' := G[A' \cup B']$ contains a matching saturating A' (Ind. Hypo.)
- $G - G'$ also satisfies Hall's condition. Why? (Consider $N_G(S \cup A')$ if $S \subset A - A'$ does not satisfy Hall's condition). $G - G'$ contains a matching saturating $A \setminus A'$. Done.

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Observation: $d_H(a) \geq 1$ for every $a \in A$ by Hall's condition.

Claim: $d_H(a) = 1$ for every $a \in A$. (By this claim, we know that the edges of H form a matching saturating A.)
Proof by contradiction:
See next slide.

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- Since $b_1 \in B_2$ and $b_2 \in B_1$, we obtain

$$\begin{aligned} |N_H(A_1 \cap A_2 \setminus \{a\}| &\leq |B_1 \cap B_2| \\ &= |B_1| + |B_2| - |B_1 \cup B_2| \\ &= |B_1| + |B_2| - |N_H(A_1 \cup A_2)| \\ &\leq |A_1| - 1 + |A_2| - 1 - |A_1 \cup A_2| \\ &= |A_1 \cap A_2 \setminus \{a\}| - 1. \end{aligned}$$

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- Since H violates Hall's condition, this contradicts with the initial assumption.

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Proof: Exercise.

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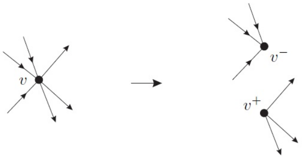

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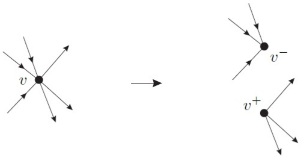

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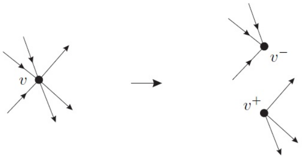

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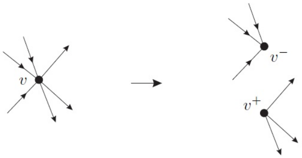

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- Since G' is a k-regular bipartite graph, by the previous corollary, G' has a perfect matching (1-factor).

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since every odd component of $G - S$ will have a factor edge between S and itself.

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Theorem (Tutte, 1947): A graph has a 1-factor if and only if Tutte's condition holds.

Corollary (Petersen, 1891): Every bridgeless (with no cut-edge) cubic (3-regular) graph has a 1-factor.
Proof: Exercise.

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Exercises

See Webpage.