CMP 694 Graph Theory
Hacettepe University

Lecture 7: Vertex Coloring and Upper Bounds

Lecturer:
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Resources:
“Introduction to Graph Theory” by Douglas B. West

Outline

Definitions

k-coloring of a graph G: A labeling $f : V(G) \implies S$, where $|S| = k$.
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chromatic number of a graph G, $\chi(G)$: The least k such that G is k-colorable.
Examples: bipartite graphs have chromatic number 2, odd cycles, Petersen graph have chromatic number 3. Why? What is the chromatic number of Q_n?

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The chromatic number of the *join* of two graphs:

$$\chi(G \vee H) = \chi(G) + \chi(H).$$

Another Product of Graphs: *Cartesian product*

The cartesian product of G and H, $G \square H$, is the graph with vertex set $V(G) \times V(H)$ specified by putting an edge between the vertices uv and $u'v'$ iff

1. $u = u'$ and $vv' \in E(H)$, or
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The chromatic number of the *cartesian product* of two graphs (Vizing, 1963, Aberth, 1964):

$$\chi(G \square H) = \max\{\chi(G), \chi(H)\}.$$

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If a graph G has a degree sequence $d_1 \ge d_2 \ge \cdots \ge d_n$, then

$$\chi(G) \le 1 + \max_i \min\{d_i, i-1\}.$$

Proof idea: Apply greedy coloring to the vertices ordered with nonincreasing degrees.

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Note: Every graph has some vertex ordering for which greedy coloring uses exactly $\chi(G)$ colors. (Exercise 33)

Color-critical (or k-critical) graphs

If $\chi(H) < \chi(G) = k$ for every proper subgraph $H \subset G$, then G is called k-critical (or color-critical).
Example: Every odd cycle is a 2-critical graph, any K_n is n-critical.

Lemma

If H is a k-critical graph, then $\delta(H) \geq k - 1$.

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Proof idea: Let H' be a k-critical subgraph of G.

$$\chi(G) - 1 = \chi(H') - 1 \leq \delta(H') \leq \max_{H \subseteq G} \delta(H).$$

Brook's Theorem

Theorem (Brooks, 1941)

If G is a connected graph other than a complete graph or an odd cycle, then $\chi(G) \leq \Delta(G)$.

Sketch of the proof: Let $k = \Delta(G)$. For $k \geq 3$, trivial for $k = 1, 2$.

- **Case 1:** G **is not** k**-regular.** Let $\deg(v_n) < k$, construct a spanning tree of G using BFS starting at v_n, label the vertices v_i with decreasing index i as they are added to the tree. Greedy algorithm uses at most k colors.

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- **Case 2: G is k-regular and has a cut-vertex:** Say x is a cut-vertex and H_1 is a component of $G - x$ and $H_2 = G - \{x\} - H_1$. Color $H_1 \cup \{x\}$ and $H_2 cup \{x\}$ separately. Permute colors in both colorings such that x has the same color in both. Done.

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- **Case 3:** G **is** k**-regular and 2-connected:** Assume some vertex v_n has neighbors v_1 and v_2, that are not adjacent, and $G - \{v_1, v_2\}$ is connected. (We show later, that this is always true.)

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Claim

Every k-regular 2-connected graph has a triple as v_1, v_2, v_n.

Proof: Since G is not complete, there are two vertices of distance 2, say v_1 and v_2. We let the common neighbor of them be v_n.

Graphs with large chromatic number

Construction (Mycielski's construction)

For an input graph G with vertices $\{v_1, \ldots, v_n\}$, a new graph G' is obtained by adding vertices $U = \{u_1, \ldots, u_n\}$ and another vertex w. The edge set og G' contains $E(G)$, the edges between u_i and $N_G(v_i)$ for all i. Moreover, let $N(w) = U$.

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From a k-chromatic triangle-free graph G, Mycielski's construction produces a $k+1$-chromatic triangle-free graph.

- U is an independent set. So, triangles could be induced by some u_i and neighbors in $N(v_i)$, contradiction, because G has no triangle.

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- Also, at least $k+1$ colors are needed. To show that start with a proper coloring of G' and obtain a proper coloring of G using less colors.

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Example: Edge-coloring of K_{2n} is a modeling of scheduling problem.

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Thus, there are two types of graphs: the ones that have edge-chromatic number $\Delta(G)$ or $\Delta(G) + 1$.