CMP 694 Graph Theory
Hacettepe University

Lecture 6: Connectivity and Menger's Theorem

Lecturer:
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Resources:
"Introduction to Graph Theory" by Douglas B. West

Some definitions on k-connectedness

Given $x, y \in V(G)$, a set $S \subseteq V(G) - \{x, y\}$ is an x, y-separator or x, y-cut if $G - S$ has no x, y-path.

Some definitions on k-connectedness

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$\kappa(x, y)$ (or $\kappa(x, y)$): the minimum size of an x, y-cut in a graph G.
$\lambda(x, y)$ (or $\lambda_G(x, y)$): the maximum size of a set of pairwise internally disjoint x, y-paths in G.

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Remark: Always, $\kappa(x, y) \geq \lambda(x, y)$. Why? See example on page 166.

Connectivity and Menger's Theorem

Theorem (Menger's Theorem)

If x and y are vertices of a graph G and $xy \notin E(G)$, then $\kappa(x, y) = \lambda(x, y)$.

Proof:

- Clearly, $\kappa(x, y) \geq \lambda(x, y)$. **Induction on** $n(G)$ to show that $\kappa(x, y) \leq \lambda(x, y)$.

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- **Inductive step:** Reading exercise.

Theorem (Menger's Theorem)

A graph G is k-connected if and only if every two vertices are connected by at least k independent paths.

Appplications of Menger's Theorem

U-fan: Given a vertex x and a set U of vertices, an x, U-fan is a set of x, U-paths such that any two of these paths have only x in common.

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Theorem (Fan Lemma, Dirac, 1960)

A graph G is k-connected if and only if it has at least $k+1$ vertices and, for every choice of $x, U \subset V(G)$ with $|U| \geq k$, it has an x, U-fan of size k.

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Remove y. These paths show an x, U-fan exists.

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- For any two vertices w and z, we find k internally disjoint w, z-paths. Thus **Menger's Thm.** implies k-connectedness of G.
- Let $U = N(z)$. There is a w, U-fan. By extending each of the k w, U-paths to z, we are done.

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Theorem (Dirac, 1960)

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- **Base step:** k**=2** If G is 2-connected, there are 2 internally disjoint x, y-paths between any two vertices x and y, whose union is a cycle containing x and y.

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- **Case:** $|V(C)| = k - 1$ Because there is an $x, V(C)$-fan, x can be added to C to obtain a larger cycle.

Proof continued

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- **Case:** $|V(C)| \geq k$ There is an $x, V(C)$-fan of size k. Let $v_1, v_2, \ldots, v_{k-1}$ be the vertices of this fan in $V(C)$.

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  V_i be the segment of $C$from v_i to v_{i+1} but not containing it.
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- By pigeonhole principle ($k-1$ segments=pigeonholes and k vertices (that are not x) of the fan $=$ pigeons) , one segment V_j contains at least two vertices.
- Say u, u' from the fan are in V_j. Replace u, u'-segment of C with the x, u-path and x, u'-path of the fan to obtain a cycle containing all of S.

Appplications of Menger's Theorem: SDR

A system of distinct representatives (SDR) for a sequence of (not necessarily distinct) sets $S_1, S_2, \ldots, S_m$ is a sequence of distinct elements $x_1, x_2, \ldots, x_m$ such that $x_i \in S_i$ for $1 \leq i \leq m$.

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Theorem (Ford-Fulkerson, 1958)

Families $\mathcal{A} = \{A_1, \ldots, A_m\}$ and $\mathcal{B} = \{B_1, \ldots, B_m\}$ have a common SDR (an SDR for both) iff

$$|(\cup_{i\in I} A_i) \cap (\cup_{j\in J} B_j)| \ge |I| + |J| - m \text{ for each pair } I, J \subseteq [m].(*)$$

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- Create a digraph G with vertices $a_1, a_2, \ldots, a_m$ and $b_1, b_2, \ldots, b_m$. In addition, add a vertex for each element in the sets and special vertices s and t.

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- The edges of G consist of $\{sa_i : A_i \in \mathcal{A}\} \cup \{b_j t : B_j \in \mathcal{B}\}$ and $\{a_i x : x \in A_i\} \cup \{x b_j : x \in B_j\}$.

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  There is a set of m pairwise internally disjoint s, t-paths if and only if there is a CSDR.

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- Let $I = \{a_i\} - R$ and $J = \{b_j\} - R$. Then, we have

$$|R| \geq |(\cup_{i \in I} A_i) \cap (\cup_{j \in J} B_j))| + (m - |I|) + (m - |J|),$$

which is always at least m because of (*).

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- See example, page 172.