CMP 694 Graph Theory
Hacettepe University

Lecture 5: Cuts and Connectivity

Lecturer:
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Resources:
“Introduction to Graph Theory” by Douglas B. West

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For a set $S \subset V(G)$, the subgraph $G[S]$ is the subgraph of G induced by S. In other words, the vertex set of the subgraph $G[S]$ is S and each edge in $G[S]$ has both of its endvertices in S.

Finding a connected subgraph of a certain order (number of vertices)

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- This enumeration of the vertices satisfy the condition we want.

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- Since there is a vertex v with $\delta(G)$ neighbors, $N(v)$ is a cut-set of G.

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- S must contain also a vertex in Q', otherwise all vertices in Q and Q' are connected to each other. Thus S contains at least k vertices. Done.

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- Let T be a vertex set that contains exactly one endpoint of each edge in F. This set is a vertex-cut of G.

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For any vertex set $S \subset V(G)$, $|[S, \bar{S}]| = [\sum_{v \in S} \deg(v)] - 2e(G[S])$. Moreover, for simple G, if $|[S, \bar{S}]| < \delta(G)$ for nonempty S, then $|S| > \delta(G)$.

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A graph G having at least three vertices is 2-connected if and only if for each pair $u, v \in V(G)$, there exist internally disjoint u, v-paths in G.

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- If $y \notin S$ and $N(y) \subset S$, again $|S| \geq k$.
- Otherwise, in $G' - S$, y and some vertices in $N(y)$ must be in the same component. This implies, S also is a vertex-cut in G and $|S| \geq k$.

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Theorem

For a graph G with at least three vertices, TFAE ("the following are equivalent") and characterize 2-connected graphs: A) G is connected and has no cut-vertex.
B) For all $x, y \in V(G)$, there are internally disjoint x, y-paths.
C) For all $x, y \in V(G)$, there is a cycle through x and y.
D) $\delta(G) \geq 1$, and every pair of edges in G lies on a common cycle.

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- $D \implies C$: Since $\delta(G) \geq 1$, no isolated vertex.
  Pick any two edges ux and vy, there is a cycle containing these edges, done. If only one edge, xy, then pick any other edge and apply to them.

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- $A \implies D$: Since G is connected, $\delta(G) \geq 1$.

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- $A \implies D$: Since G is connected, $\delta(G) \geq 1$.
- Consider any two edges uv and xy. Add to G a new vertex w with neighbors u and v. Add another vertex z to G with neighbors x and y. Call this new graph G'.

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- Consider any two edges uv and xy. Add to G a new vertex w with neighbors u and v. Add another vertex z to G with neighbors x and y. Call this new graph G'.
- By expansion lemma, G' is 2-connected. Since $A \iff C$, w and z lie on a cycle. Remove w and z from C and add the edges uv and xy, done.

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Ear decomposition: An ear of a graph G is a maximal path whose internal vertices have degree 2 in G. An ear decomposition of G is a decomposition $P_0, \ldots, P_k$ such that P_0 is a cycle and P_i for $i \geq 1$ is an ear of $P_0 \cup \cdots \cup P_i$.

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Corollary

If G is 2-connected, then the graph G' obtained by sibdividing an edge is 2-connected.

Ear decomposition: An ear of a graph G is a maximal path whose internal vertices have degree 2 in G. An ear decomposition of G is a decomposition $P_0, \dots, P_k$ such that P_0 is a cycle and P_i for $i \geq 1$ is an ear of $P_0 \cup \cdots \cup P_i$.

Theorem (Whitney, 1932)

A graph is 2-connnected iff it has an ear decomposition. Moreover, every cycle in a 2-connected graph is the initial cycle in some ear decomposition.

A similar decomposition exists also for 2-edge-connected graphs (see the book).