CMP 694 Graph Theory
Hacettepe University

Lecture 3: Matchings and Covers

Lecturer:
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Resources:
“Introduction to Graph Theory” by Douglas B. West

Matching and Perfect Matching

A matching is a set of independent edges in a graph. A matching is

- maximal if it cannot be made larger by adding any more edges.
- maximum if it is maximal and the largest possible matching in a graph
- perfect if it contains all vertices in the graph (only possible, if vertex number is even).

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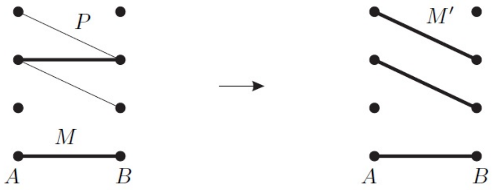

Figure: Augmenting the matching M by the alternating path P

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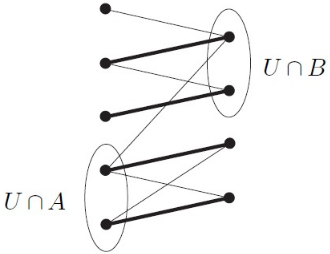

Figure: The vertex cover U

- Let ab be an edge; show that either a or b lies in U.

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Theorem (Hall, 1935): G contains a matching that saturates A if and only if $|N(S)| \geq |S|$ for all $S \subset A$.

First proof of Hall's Theorem

Proof by induction:

- Apply induction on $|A|$. For $|A| = 1$, clearly the theorem holds. Let $|A| \geq 2$ and assume that Hall's condition is sufficient of a mathing that saturates A when $|A|$ is smaller.

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  $$|N_{G'}(S)| \geq |N_G(S)| - 1 \geq |S|$$

  for every $S \subset A \setminus \{a\}$.
- G' contains a matching that saturates $A \setminus \{a\}$ by inductive hypothesis, this matching together with ab is a matching of G.

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  (Consider $N_G(S \cup A')$ if $S \subset A - A'$ does not satisfy Hall's condition).

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- $G - G'$ also satisfies Hall's condition. Why?
  (Consider $N_G(S \cup A')$ if $S \subset A - A'$ does not satisfy Hall's condition). $G - G'$ contains a matching saturating $A \setminus A'$. Done.

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Observation: $d_H(a) \geq 1$ for every $a \in A$ by Hall's condition.

Claim: $d_H(a) = 1$ for every $a \in A$. (By this claim, we know that the edges of H form a matching saturating A.)
Proof by contradiction:
See next slide.

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- Since $b_1 \in B_2$ and $b_2 \in B_1$, we obtain

$$\begin{aligned}
|N_H(A_1 \cap A_2 \setminus \{a\}| &\leq |B_1 \cap B_2| \\
&= |B_1| + |B_2| - |B_1 \cup B_2| \\
&= |B_1| + |B_2| - |N_H(A_1 \cup A_2)| \\
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- Since H violates Hall's condition, this contradicts with the initial assumption.

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- Since H and H' are vertex-disjoint, these the union of these two mathings is a matching of G. Done.