CMP 694 Graph Theory
Hacettepe University

Lecture 2: Trees

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Resources:
http://www.inf.ed.ac.uk/teaching/courses/dmmr
"Introduction to Graph Theory" by Douglas B. West

A **tree** is a connected simple undirected graph with no simple circuits.
A **forest** is a (not necessarily connected) simple undirected graph with no simple circuits.

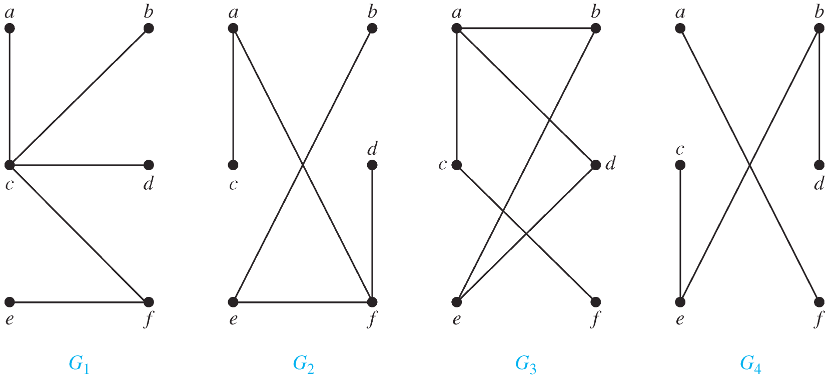

FIGURE 2 **Examples of Trees and Graphs That Are Not Trees.**

Some important facts about trees

Theorem 1: A graph G is a tree if and only if there is a unique simple (and tidy) path between any two vertices of G.

Proof: On the board. (Next slide provides written proof.) □

Theorem 2: Every tree, $T = (V, E)$ with $|V| \geq 2$, has at least two vertices that have degree $= 1$.

Proof: Take any **longest** simple path $x_0 \ldots x_m$ in T. Both x_0 and x_m must have degree 1: otherwise there's a longer path in T. □

Theorem 3: Every tree with n vertices has exactly $n - 1$ edges.

Proof: On the board. By induction on n. □

Proof of Theorem 1 about Trees

Suppose there are two distinct simple paths between vertices $u, v \in V$: $x_0 x_1 x_2 \ldots x_n$ and $y_0 y_1 y_2 \ldots y_m$.
Firstly, there must be some $i \geq 1$, such that $\forall 0 \leq k < i$, $x_k = y_k$, but such that $x_i \neq y_i$. (Why is this so?)
Furthermore, there must be a smallest $j \geq i$, such that either x_j appears in $y_i, \ldots, y_m$, or such that y_j appears in $x_i \ldots x_n$.
Suppose, without loss of generality, that this holds for some smallest $j \geq i$ and x_j. Then $x_j = y_r$, for some smallest $r \geq i$.
We claim that then the path $x_{i-1} x_i \ldots x_j y_{r-1} y_{r-2} \ldots y_i y_{i-1}$ must form a simple circuit, which contradicts the fact that G is a tree.
Note that by assumption $x_{i-1} = y_{i-1}$, so this is a circuit.
Furthermore, it is simple, because its edges are a disjoint union of edges from the x and y paths, because by construction none of the vertices $x_i, \ldots, x_j$ occur in $y_i \ldots y_{r-1}$, and $x_i \neq y_i$. □

Different Definitions of Trees

Theorem

The following statements are equivalent for a graph T:

1. *T is a tree;*
2. *Any two vertices of T are connected by a unique path in T;*
3. *T is minimally connected, that is, T is connected but $T - e$ is disconnected for every edge e in T;*
4. *T is maximally acylic, that is T contains no cycle, but $T + xy$ does, for any two non-adjacent vertices x, y in T.*

Observations on Trees

Corollary: The vertices of a tree can always be enumerated, say as $v_1, \ldots, v_n$ so that every v_i with $i \geq 2$ has a unique neighbor in $\{v_1, \ldots, v_{i-1}\}$.

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Proof: Exercise.

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Proof: Exercise.

Corollary: If T is a tree and G is any graph with $\delta(G) \geq |T| - 1$, then $T \subset G$, that is G has a subgraph isomorphic to T.
Proof Construct T by using a greedy algorithm.