BIL694-Lecture 1: Introduction to Graphs

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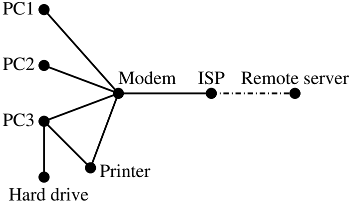
Resources for the presentation:
http://www.math.ucsd.edu/ gptesler/184a/calendar.html
http://www.inf.ed.ac.uk/teaching/courses/dmmr/

Outline

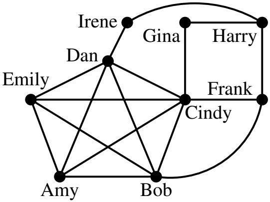
Outline

Graphs

Computer network



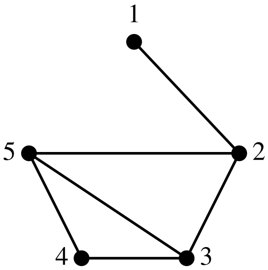
Friends



We have a network of items and connections between them. Examples:

- Telephone networks, computer networks
- Transportation networks (bus/subway/train/plane)
- Social networks
- Family trees, evolutionary trees
- Molecular graphs (atoms and chemical bonds)
- Various data structures in Computer Science

Graphs



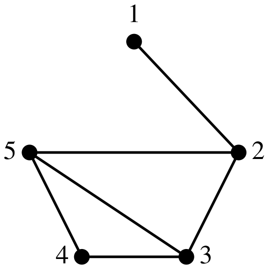
- The dots are called *vertices* or *nodes* (singular: vertex, node)

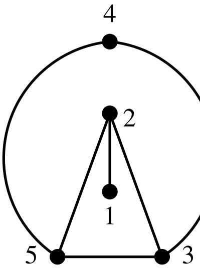
$$V = \text{set of vertices} = \{1, 2, 3, 4, 5\}$$

- The connections between vertices are called *edges*.
- Represent an edge as a set $\{i, j\}$ of two vertices.
  E.g., the edge between 2 and 5 is $\{2, 5\} = \{5, 2\}$.

$$E = \text{set of edges} = \Big\{\{1, 2\}, \{2, 3\}, \{2, 5\}, \{3, 4\}, \{3, 5\}, \{4, 5\}\Big\}$$

Simple graphs

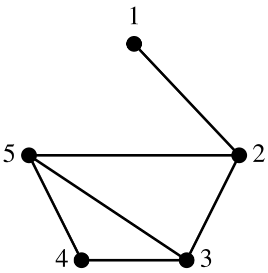




A *simple graph* is $G = (V, E)$:

- V is the set of vertices.
  It can be any set; $\{1, \ldots, n\}$ is just an example.
- E is the set of edges, of form $\{u, v\}$, where $u, v \in V$ and $u \neq v$.
  Every pair of vertices has either 0 or 1 edges between them.
- The drawings above represent the same abstract graph since they have the same V and E, even though the drawings look different.

Degrees



The *degree* of a vertex is the number of edges on it.

$$d(1) = 1 \qquad d(2) = 3 \qquad d(3) = 3 \qquad d(4) = 2 \qquad d(5) = 3$$

Sum of degrees $= 1 + 3 + 3 + 2 + 3 = 12$
Number of edges $= 6$

Sum of degrees

Theorem

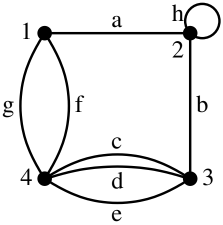
The sum of degrees of all vertices is twice the number of edges:

$$\sum_{v\in V} d(v) = 2\,|E|$$

Proof.

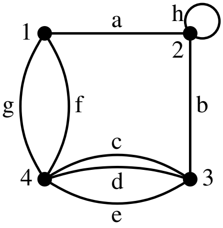
- Let $S = \{\, (v,e) \;:\; v \in V,\;\; e \in E,\;\; \text{vertex } v \text{ is in edge } e \,\}$
- **Count $|S|$ by vertices:** Each vertex v is contained in $d(v)$ edges, so
$$|S| = \sum_{v\in V} d(v).$$
- **Count $|S|$ by edges:** Each edge has two vertices, so
$$|S| = \sum_{e\in E} 2 = 2\,|E|\;.$$

Multigraphs



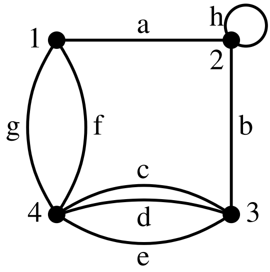
- Some networks have *multiple edges* between two vertices. Notation $\{3, 4\}$ is ambiguous, so write labels on the edges: c, d, e.
- There can be an edge from a vertex to itself, called a *loop* (such as h above). A loop has one vertex, so $\{2, 2\} = \{2\}$.
- A simple graph does not have multiple edges or loops.

Multigraphs



- Computer network with multiple connections between machines.
- Transportation network with multiple routes between stations.
- **But:** A graph of Facebook friends is a simple graph. It does not have multiple edges, since you're either friends or you're not. Also, you cannot be your own Facebook friend, so no loops.

Multigraphs



$$V = \{1, 2, 3, 4\}$$
$$E = \{a, b, c, d, e, f, g, h\}$$

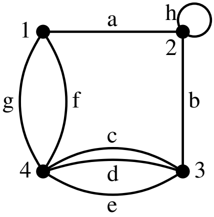
$$\phi(a) = \{1, 2\}$$
$$\phi(b) = \{2, 3\}$$
$$\phi(c) = \phi(d) = \phi(e) = \{3, 4\}$$
$$\phi(f) = \phi(g) = \{1, 4\}$$
$$\phi(h) = \{2\}$$

A *multigraph* is $G = (V, E, \phi)$, where:

- V is the set of vertices. It can be any set.
- E is the set of edge labels (with a unique label for each edge).
- $\phi : E \to \{\,\{u, v\} \;:\; u, v \in V\,\}$
  is a function from the edge labels to the pairs of vertices.
  $\phi(L) = \{u, v\}$ means the edge with label L connects u and v.

Adjacency matrix of a multigraph

- Let $n = |V|$
- The *adjacency matrix* of a multigraph is an $n \times n$ matrix $A = (a_{uv})$. Entry a_{uv} is the number of edges between vertices $u, v \in V$.



$$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 2 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 3 \\ 2 & 0 & 3 & 0 \end{bmatrix} \end{array}$$

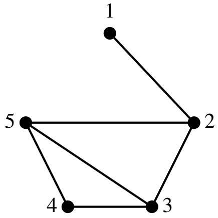
- $a_{uv} = a_{vu}$ for all vertices u, v. Thus, A is a *symmetric matrix* ($A = A^T$).
- The sum of entries in row u is the degree of u.
- **Technicality:** A loop on vertex v counts as
  - 1 edge in E,
  - degree 2 in $d(v)$ and in a_{vv} (it touches vertex v twice),

  With these rules, graphs with loops also satisfy $\sum_{v \in V} d(v) = 2|E|$.

Adjacency matrix of a simple graph

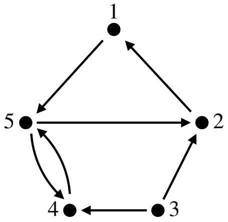
In a simple graph:

- All entries of the adjacency matrix are 0 or 1 (since there either is or is not an edge between each pair of vertices).
- The diagonal is all 0's (since there are no loops).



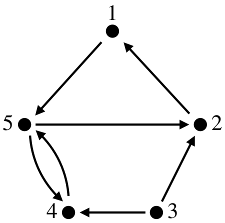
$$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Directed graph (a.k.a. digraph)



- A *directed edge* is a connection with a direction.
- One-way transportation routes.
- Broadcast TV and satellite TV are one-way connections from the broadcaster to your antenna.
- Familiy tree: parent $\rightarrow$ child
- An unrequited Facebook friend request.

Directed graph (a.k.a. digraph)

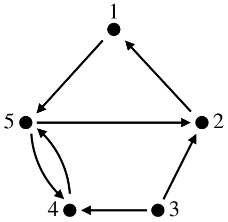


$$V = \{1, 2, 3, 4, 5\}$$

$$E = \{(1,5), (2,1), (3,2), (3,4), (4,5), (5,2), (5,4)\}$$

- Represent a directed edge $u \to v$ by an ordered pair (u, v).
  E.g., $3 \to 2$ is $(3, 2)$, but we do not have $2 \to 3$, which is $(2, 3)$.
- A directed graph is *simple* if each (u, v) occurs at most once, and there are no loops.
  - Represent it as $G = (V, E)$.
  - V is a set of vertices. It can be any set.
  - E is the set of edges. Each edge has form (u, v) with $u, v \in V$, $u \neq v$.
  - It is permissible to have both $(4, 5)$ and $(5, 4)$, since they are distinct.

Degrees in a directed graph

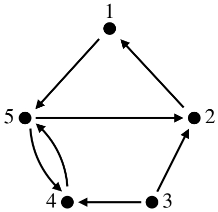


- For a vertex v, the *indegree* is the number of edges going into v, and the *outdegree* is the number of edges going out from v.

v	$\text{indegree}(v)$	$\text{outdegree}(v)$
1	1	1
2	2	1
3	0	2
4	2	1
5	2	2
Total	7	7

- The sum of indegrees is $|E|$ and the sum of outdegrees is $|E|$.

Adjacency matrix of a directed graph



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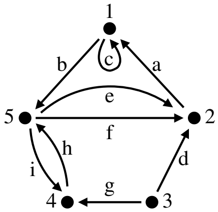
- Let $n = |V|$
- The *adjacency matrix* of a directed graph is an $n \times n$ matrix $A = (a_{uv})$ with $u, v \in V$.
- Entry a_{uv} is the number of edges directed from u to v.
- a_{uv} and a_{vu} are not necessarily equal, so A is usually not symmetric.
- The sum of entries in row u is the outdegree of u.
  The sum of entries in column v is the indegree of v.

Directed multigraph



$$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

$V = \{1, \ldots, 5\}$	$\phi(a) = (2, 1)$	$\phi(d) = (3, 2)$	$\phi(g) = (3, 4)$
$E = \{a, \ldots, i\}$	$\phi(b) = (1, 5)$	$\phi(e) = (5, 2)$	$\phi(h) = (4, 5)$
	$\phi(c) = (1, 1)$	$\phi(f) = (5, 2)$	$\phi(i) = (5, 4)$

- A directed multigraph may have loops and multiple edges.
  - Represent it as $G = (V, E, \phi)$.
  - Name the edges with labels. Let E be the set of the labels.
  - $\phi(L) = (u, v)$ means the edge with label L goes from u to v.
- **Technicality:** A loop counts once in indegree, outdegree, and a_{vv}.

Outline

Isomorphism of Graphs

Definition: Two (undirected) graphs *G1* = (*V1*, *E*1) and *G*2 = (*V*2, *E*2) are *isomorphic* if there is a bijection, $f: V_1 \rightarrow V_2$, with the property that for all vertices $a, b \in V_1$

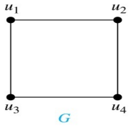
$$\{a, b\} \in E_1 \quad \text{if and only if} \quad \{f(a), f(b)\} \in E_2$$

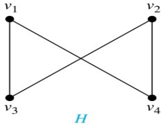
Such a function *f* is called an *isomorphism*.

Intuitively, isomorphic graphs are “THE SAME”, except for “renamed” vertices.

Isomorphism of Graphs (*cont.*)

Example: Show that the graphs $G = (V, E)$ and $H = (W, F)$ are isomorphic.





Solution: The function f with $f(u1) = v1$, $f(u2) = v4$, $f(u3) = v3$, and $f(u4) = v2$ is a one-to-one correspondence between V and W.

Isomorphism of Graphs (*cont.*)

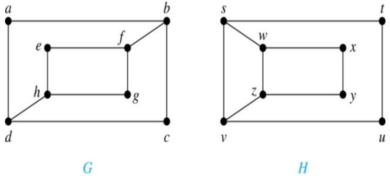
It is difficult to determine whether two graphs are isomorphic by brute force: there are $n!$ bijections between vertices of two n-vertex graphs.

Often, we can show two graphs are not isomorphic by finding a property that only one of the two graphs has. Such a property is called *graph invariant*:

- e.g., number of vertices of given degree, the degree sequence (list of the degrees), ...

Isomorphism of Graphs (*cont.*)

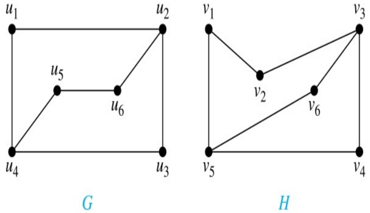
Example: Are these graphs are isomorphic?



Solution: No! Since *deg*(*a*) = 2 in *G*, *a* must correspond to *t*, *u*, *x*, or *y*, since these are the vertices of degree 2 in H. But each of these vertices is adjacent to another vertex of degree 2 in *H*, which is not true for *a* in *G*. So, G and H can not be isomorphic.

Isomorphism of Graphs (*cont.*)

Example: Determine whether these two graphs are isomorphic.



Solution: The function *f* is defined by: $f(u1) = v6$, $f(u2) = v3$, $f(u3) = v4$, $f(u4) = v5$, $f(u5) = v1$, and $f(u6) = v2$ is a bijection.

Algorithms for Graph Isomorphism

- The best algorithms known for determining whether two graphs are isomorphic have exponential worst-case time complexity (in the number of vertices of the graphs).
- However, there are algorithms with good time complexity in many practical cases.
- See, e.g., a publicly available software called NAUTY for graph isomorphism.

Applications of Graph Isomorphism

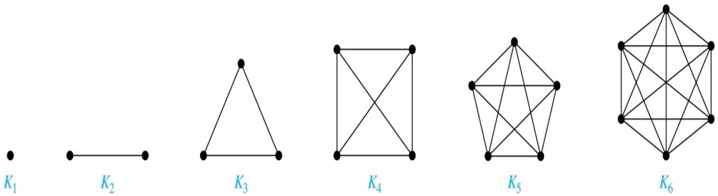
The question whether graphs are isomorphic plays an important role in applications of graph theory. For example:

Chemists use molecular graphs to model chemical compounds. Vertices represent atoms and edges represent chemical bonds. When a new compound is synthesized, a database of molecular graphs is checked to determine whether the new compound is isomorphic to the graph of an already known one.

Outline

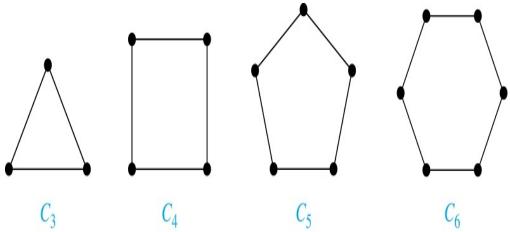
Special Types of Graphs: Complete Graphs

A *complete graph on n vertices*, denoted by K_n, is the simple graph that contains exactly one edge between each pair of distinct vertices.



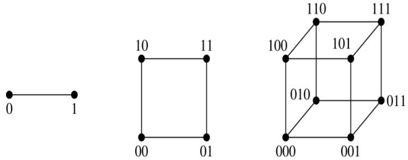
Special Types of Graphs: Cycles

A *cycle* C_n for *n* ≥ 3 consists of *n* vertices *v*1, *v*2 ,⋯ , *v*n, and edges {*v*1, *v*2}, {*v*2, *v*3} ,⋯ , {*vn*-1, *vn*}, {*vn*, *v*1}.



Special Types of Simple Graphs: *n*-Cubes

An *n-dimensional hypercube*, or *n-cube*, is a graph with 2^n vertices representing all bit strings of length *n*, where there is an edge between two vertices if and only if they differ in exactly one bit position.



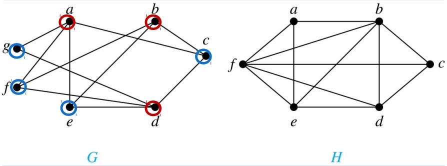
Q_1 Q_2 Q_3

Bipartite Graphs

Definition:

An equivalent definition of a bipartite graph is one where it is possible to color the vertices either red or blue so that no two adjacent vertices are the same color.

G is bipartite

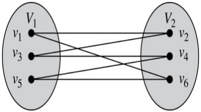


H is not bipartite: if we color *a* red, then its neighbors *f* and *b* must be blue. But f and b are adjacent.

Bipartite Graphs (*continued*)

Example: Show that C_6 is bipartite.

Solution: Partition the vertex set into *V*1 = {*v*1, *v*3, *v*5} and *V*2 = {*v*2, *v*4, *v*6}:



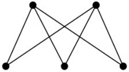
Example: Show that *C_3* is not bipartite.

Solution: If we partition vertices of *C*_3 into two nonempty sets, one set must contain two vertices. But every vertex is connected to every other. So, the two vertices in the same partition are connected. Hence, *C*_3 is not bipartite.

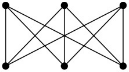
Complete Bipartite Graphs

Definition: A *complete bipartite graph* is a graph that has its vertex set partitioned $K_{m,n}$ into two subsets *V*_1 of size *m* and *V*_2 of size *n* such that there is an edge from every vertex in *V*_1 to every vertex in *V*_2.

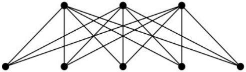
Examples:



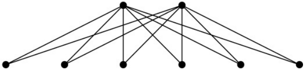
$K_{2,3}$



$K_{3,3}$



$K_{3,5}$

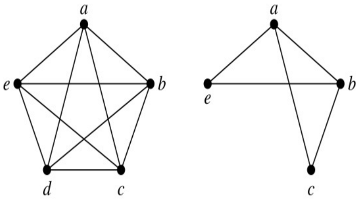


$K_{2,6}$

Subgraphs

Definition: A *subgraph of a graph* *G* = (*V*,*E*) is a graph (*W*,*F*), where $W \subseteq V$ and $F \subseteq E$. A subgraph *H* of *G* is a proper subgraph of *G* if *H* ≠ *G*.

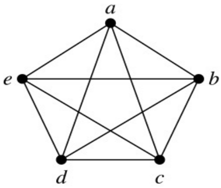
Example: here is K_5 and one of its (proper) subgraphs:

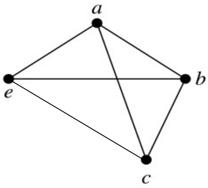


Induced Subgraphs

Definition: Let *G* = (*V*, *E*) be a graph. The *subgraph induced* by a subset *W* of the vertex set *V* is the graph H= (*W*,*F*), whose edge set *F* contains an edge in *E* if and only if both endpoints are in *W*.

Example: Here is *K*_5 and its induced subgraph induced by *W* = {*a*,*b*,*c*,*e*}.





Bipartite Graphs

Theorem (König, 1936)

A graph is bipartite if and only if it has no odd cycle.

Proof

Necessity: If G is bipartite, clearly every cycle has even length.

Sufficiency:

- Assume that G has no odd cycle, show that G is bipartite.
- Let $u \in V(G)$ and for each $v \in V(G)$, let $f(v)$ be the minimum length of a u, v-path (finite, assuming G is connnected).
- $X := \{v \in V(G) : f(v) \text{is even}\}$, $\qquad Y := \{v \in V(G) : f(v) \text{is odd}\}$.

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- $X := \{v \in V(G) : f(v) \text{is even}\}, \qquad Y := \{v \in V(G) : f(v) \text{is odd}\}.$
- Any edge in $vw \in G[X]$, together with the shortest u, v- and u, w-paths create an odd walk, call it W. (Same if vw is an edge in $G[Y]$)

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- Any edge in $vw \in G[X]$, together with the shortest u, v- and u, w-paths create an odd walk, call it W. (Same if vw is an edge in $G[Y]$)
- And any odd walk contains an odd cycle (by Lemma 1.2.15 in the book). Contradiction!

Bipartite Graphs

Theorem (König, 1936)

A graph is bipartite if and only if it has no odd cycle.

Proof

Necessity: If G is bipartite, clearly every cycle has even length.

Sufficiency:

- Assume that G has no odd cycle, show that G is bipartite.
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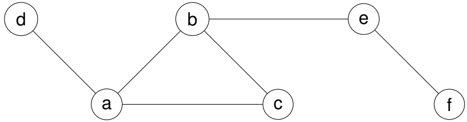
- Therefore, X and Y are independent sets.
- Same ideas can be applied for each component, if G had more than one component.

Outline

Connectness in undirected graphs

Definition: An undirected graph $G = (V, E)$ is called connected, if there is a path between every pair of distinct vertices.
It is called disconnnected otherwise.



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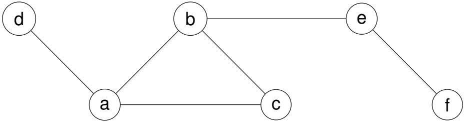
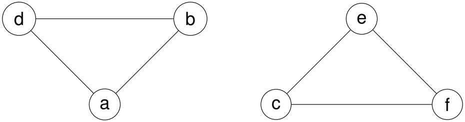
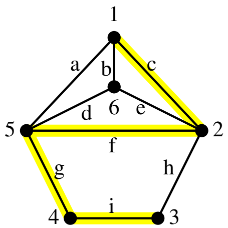


This graph is connected



This graph is **not** connected

Walks



- Trace along edges from vertex x to y, without lifting your pen.
- The walk in yellow is represented as a sequence of edges c, f, g, i, or a sequence of vertices $1, 2, 5, 4, 3$.
- A *walk* from vertex x to y is a sequence of edges, each connected to the next by a vertex:

  $$e_1 = \{x, v_1\} \quad e_2 = \{v_1, v_2\} \quad e_3 = \{v_2, v_3\} \quad \cdots \quad e_k = \{v_{k-1}, y\}$$

  In a directed graph, edge directions must be respected:

  $$e_1 = (x, v_1) \quad e_2 = (v_1, v_2) \quad e_3 = (v_2, v_3) \quad \cdots \quad e_k = (v_{k-1}, y)$$

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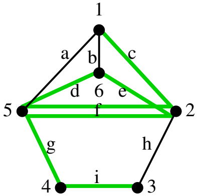
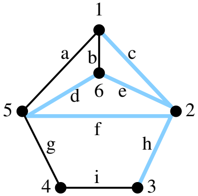
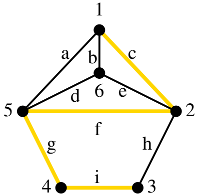


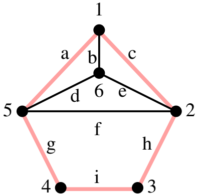
Walk



Trail



Path



Cycle

- In a *walk*, edges and vertices may be re-used.
- A *trail* is a walk with all edges distinct.
- A *path* is a walk with all vertices and edges distinct.
- A walk/trail/path is *open* if the start and end vertices are different, and *closed* if they are the same (this is allowed in a closed path, but no other vertices may be repeated).
- A *cycle* is a closed path.

Different Definitions of Trees

A tree is a simple connected graph that contains no cycle.

Theorem

The following statements are equivalent for a graph T:

1. *T is a tree;*
2. *Any two vertices of T are connected by a unique path in T;*
3. *T is minimally connected, that is, T is connected but $T - e$ is disconnected for every edge e in T;*
4. *T is maximally acylic, that is T contains no cycle, but $T + xy$ does, for any two non-adjacent vertices x, y in T.*

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Corollary: Every tree has at least two vertices with degree 1.

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Proof: Take any longest simple path $x_0, \ldots, x_m$ in T. Both x_0 and x_m must have degree 1: otherwise there is s a longer path in T.

Corollary: The vertices of a tree can always be enumerated, say as $v_1, \ldots, v_n$ so that every v_i with $i \geq 2$ has a unique neighbor in $\{v_1, \ldots, v_{i-1}\}$.

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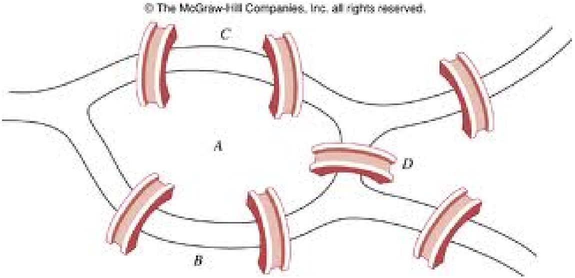
Corollary: If T is a tree and G is any graph with $\delta(G) \geq |T| - 1$, then $T \subset G$, that is G has a subgraph isomorphic to T.
Proof Construct T by using a greedy algorithm.

The Königsberg Bridge Problem

Leonard Euler (1707-1783) was asked to solve the following:



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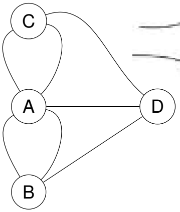

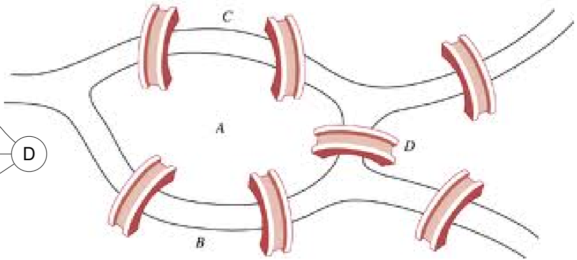

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Euler (in 1736) used “graph theory” to answer this question.

Characterization of Eulerian Graphs

Theorem

A connnected graph G is Eulerian if and only if all of its vertices have even degree.

Proof

Necessity: Add direction to each edge as it is visited along the Eulerian circuit. Since each vertex has indegree equal to its outdegree, the degree (undirected) of each vertex is even.

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- So, pick a cycle in one component, call it C. Let $G' := G - C$.
- By inductive hypothesis, every component of G' has an Euler circuit D.
- D, together with C is an Euler circuit of G.

Conclusions

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Every even graph (all degree even) decomposes into cycles.

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Proposition

If for a graph G, $\delta(G) \geq k$, then G contains a P_{k+1}. If $k \geq 2$, then G also contains a cycle of length at least $k+1$.