BIL694-Lecture 6-7: Counting

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Resources for the presentation:

"Extremal Combinatorics with Applications in Computer Science" by S. Jukna

Outline

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Theorem (Binomial Theorem)

Let n be a positive integer. Then for all x and y,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Proof.

Multiply $(x+y)\dots(x+y)$ n times and consider how many times x^k and y^k will appear in each term for $k = 0, 1, \dots, n$. □

$(n)_k = n(n-1)\dots(n-k+1)$ is the number of ordered strings with k elements chosen from n different elements.
Another way to count the number of these strings is $\binom{n}{k}k!$,
since there are $\binom{n}{k}$ ways to pick k elements and $k!$ ways to order these elements. Thus,

$$(n)_k = \binom{n}{k} k! = \frac{n!}{(n-k)!k!} k! = \frac{n!}{(n-k)!}.$$

Combinatorial Equalities and Proving Them Combinatorially

- $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

  Proof: Plug in $x = y = 1$ in the binomial theorem.

- $\binom{n}{n-k} = \binom{n}{k}$

  Mathematical proof: $\frac{n!}{(n-(n-k))!(n-k)!} = \frac{n!}{(n-k)!k!}$.

  Combinatorial proof: Each k-subset is uniquely determined by its complement in an n-set.
  Therefore, the number of ways to choose a k-set from n distinct elements is the same as the number of ways to choose an $(n-k)$-set from n distinct elements.

- *Pascal Triangle*
  $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

  Mathematical proof: *Exercise*
  Combinatorial proof: *Exercise*

How fast does $\binom{n}{k}$ grow with k?

Proposition

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \quad \textit{and} \quad \sum_{i=0}^{k} \binom{n}{i} \leq \left(\frac{en}{k}\right)^k.$$

Lower Bound:
$\left(\frac{n}{k}\right)^k = \frac{n}{k} \cdot \frac{n}{k} \cdots \frac{n}{k} \leq \frac{n}{k}\frac{n-1}{k-1} \cdots \frac{n-k+1}{1} = \binom{n}{k}.$

Upper Bound:
Recall that $e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots$ for any real number t. Thus

$$e^t < 1 + t.$$

By using the binomial theorem and above and for $t = k/n$, we obtain

$$\sum_{i=0}^{k} \binom{n}{i} \leq \sum_{i=0}^{k} \binom{n}{i} \frac{t^i}{t^k} = \frac{(1+t)^n}{t^k} \leq \left(\frac{en}{k}\right)^k.$$

Note: Tighter asymptotic (for sufficiently large n) estimates can be obtained by using the Stirling Formula:

$$n! = \left(\frac{n}{e}\right)^n \sqrt{2\pi n} e^{\alpha_n}, \quad \text{where} \qquad 1/(12n+1) < \alpha_n < 1/(12n).$$

Exercises, for more check Jukna's book

- In how many ways can we distribute k balls to n boxes so that each box has at most one ball?
- Prove **combinatorially** that
$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}.$$
Hint: Count in two ways the number of pairs (x, M), where M is a k-element subset of $\{1, \ldots, n\}$ and $x \in M$.
- Prove combinatorially that
$$\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}.$$
Hint: Count in two ways the number of pairs (x, M) with $x \in M \subset \{1, \ldots, n\}$.
- Prove the above statement mathematically.
- Use Pascal Triangle to show that
$$\sum_{i=0}^{r} \binom{n+i-1}{i} = \binom{n+r}{r}.$$

Outline

Proposition

The number of integer solutions to the equation

$$x_1 + \cdots + x_n = r$$

under the condition that $x_i \geq 0$ *for all* $i = 1, \ldots, n$*, is* $\binom{n+r-1}{r}$.

Exercise

Let $k \geq 2n$*. In how many ways can we distribute* k *sweets to* n *children, if each child is supposed to get at least 2 of them?*

Outline

Partitions

Definition

A partition of n objects is a collection of its mutually disjoint subsets, called parts, whose union gives the whole set.
Let $S(n; k_1, k_2, \dots, k_n)$ denote the number of all partitions of n distinct objects with k_i i-element parts, which implies $k_1 + 2k_2 + \dots + nk_n = n$.

Proposition

$$S(n; k_1, k_2, \dots, k_n) = \frac{n!}{k_1! \dots k_n! (1!)^{k_1} \dots (n!)^{k_n}}.$$

Exercise: 1.23 (Jukna)

Outline

Double Counting

The double counting principle states the following fact: if the elements of a set are counted in two different ways, then the answers are obviously the same.

Example:

Let M be an $m \times n$ matrix with entries 0 and 1. Then the total number of 1's added over the columns is the same as the total number of 1's added over the rows.

Example (Handshaking Lemma)

At a party, the number of guests who shake hands an odd number of times is even.

Reading exercise: Read the proof.

Proposition

Let $\mathcal{F}$ be a family of subsets of some set X. Then,

$$\sum_{x \in X} d(x) = \sum_{A \in \mathcal{F}} |A|.$$

Proof: Consider the incidence matrix with rows representing elements of X and columns representing the members of $\mathcal{F}$.

Theorem (Euler, 1736)

In every graph, the sum of the degrees of its vertices is two times the number of its edges, and hence, is even.

Outline

The Averaging Principle

The Averaging Principle says that every set of numbers must contain a number at least as large as the average and a number at least as small as the average.

Observation: Every graph on n vertices with fewer than $n-1$ edges is disconnected.

Proof: Induction on n.

For $n=1$, trivial.

By averaging, there is a vertex with degree less than 2.

A real valued function $f(x)$ is convex if

$$f(\lambda a + (1-\lambda)b) \leq \lambda f(a) + (1-\lambda)f(b),$$

for any $0 \leq lambda \leq 1$. (The graph of a convex function in the book.)

Geometrical meaning: If we draw a line ℓ through the points $(a, f(a))$ and $(b, f(b))$, then the graph of the curve $f(z)$ must lie below of $\ell(z)$ for $z \in [a, b]$.

Theorem (Jensen's Inequality)

If $0 \leq \lambda_i \leq 1$, $\sum_{i=1}^{n} \lambda_i = 1$ and f is convex, then

$$f\left(\sum_{i=1}^{n} \lambda_i x_i\right) \leq \sum_{i=1}^{n} \lambda_i f(x_i)$$

Corollary (Cauchy-Schwarz Inequality)

If $a_1, \ldots, a_n$ are non-negative, then

$$\frac{1}{n}\left(\sum_{i=1}^{n} a_i\right)^2 \leq \sum_{i=1}^{n} a_i^2.$$

Proof: Let $f(x) = x^2$ and $\lambda_i = 1/n$ for each i in Jensen's inequality.

Geometric Mean vs. Arithmetic Mean

Corollary

If $a_1, \ldots, a_n$ are non-negative, then

$$(\Pi_{i=1}^{n} a_i)^{1/n} \leq \frac{1}{n} \sum_{i=1}^{n} a_i.$$

Thus, the arithmetic mean is at least the geometric mean.

Proof: Let $f(x) = 2^x$ and $\lambda_i = 1/n$, $x_i = \log_2 a_i$ for each i in Jensen's inequality.

Outline

The Inclusion-Exclusion Principle

In general, given n subsets $A_1, \ldots, A_n$ of a set X, we want to calculate $|A_1 \cup \cdots \cup A_n|$, the number of elements in this union.

Proposition (Inclusion-Exclusion Principle)

Let $A_1, \ldots, A_n$ be subsets of $X = A_\emptyset$. Then the number of elements of X in the complement of $A_1 \cup \cdots \cup A_n$ is

$$\sum_{I \subset \{1,\ldots,n\}} (-1)^{|I|} |A_I|, \quad \textit{where} \;\; A_I := \cap_{i \in I} A_i.$$

Proof:

- The sum is a linear combination of cardinalities (sizes) of sets A_I with coeffficients +1 and -1.
- Suppose that an element $x \in X$ is in none of the A_i's, then the only time x is counted is when $I = \emptyset$.
- Otherwise, $x \in A_I$ precisely when $I \subset J$. Thus the contrribution of x is

$$\sum_{I \subset J} (-1)^{|I|} = \sum_{i=0}^{|J|} \binom{|J|}{i} (-1)^i = (1-1)^{|J|} = 0.$$

Proposition

Let $A_1, \ldots, A_n$ *be subsets of* $X = A_\emptyset$. *Then,*

$$|A_1 \cup \cdots \cup A_n| = \sum_{\emptyset \neq I \subset \{1,\ldots,n\}} (-1)^{|I|+1} |A_I|, \quad \textit{where} \ \ A_I := \cap_{i \in I} A_i.$$

Proof: Subtract the number in the previous proposition from $|X|$.

A derangement is a permutation which fixes none of its elements.

There are exactly $n!$ permutations. How many of them are derangements?

Proposition

The number of derangements of $\{1, \ldots, n\}$ *is*

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)! = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}.$$

Proof: Apply inclusion-exclusion principle.