NAME: SOLUTIONS

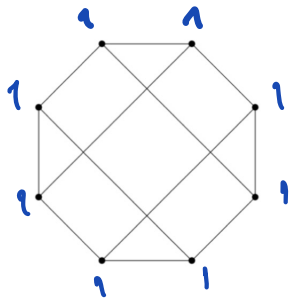
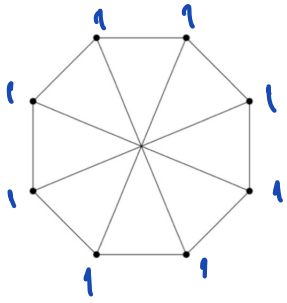
STUDENT ID:

BBM462 Final Exam
June, 12, 2023, 15:00-16:30

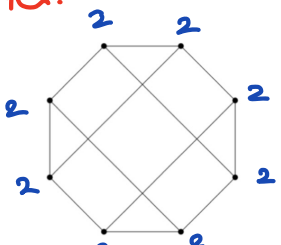
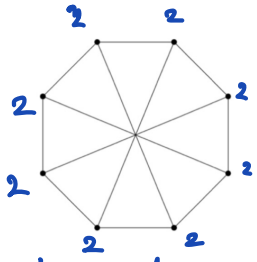
Question:	1	2	3	4	5	Total
Points:	20	20	18	20	22	100
Score:						

1. (20 points) Measure the similarity of the graphs below using the Weisfeiler-Lehman kernel using 4 iterations. Show all your work to receive full credit.

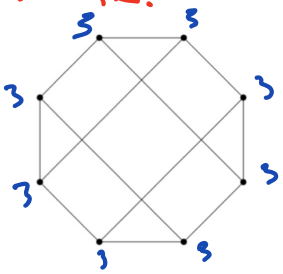
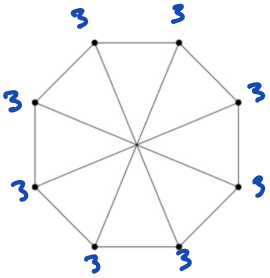
2: 1,111
3: 2,222
4: 3,333
5: 4,444

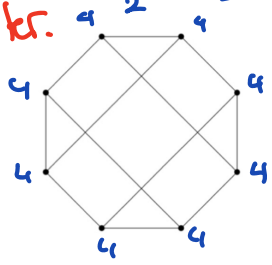
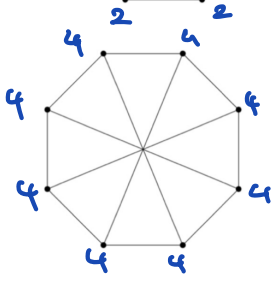
1st iter.

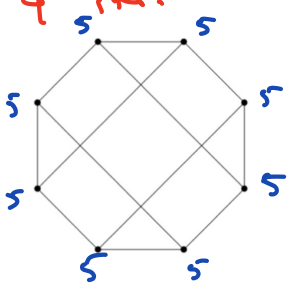
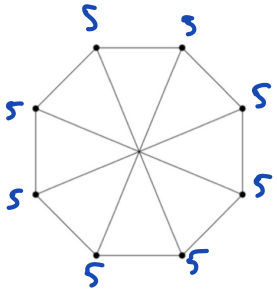
2nd iter.

3rd iter.

4th iter.

$\phi(G_1) = [8, 8, 8, 8, 8]$
$\phi(G_2) = \phi(G_1)$
$Ker(G_1, G_2) = \phi(G_1) \cdot \phi(G_2)$

2. (20 points) Calculate the degree, closeness and betweenness centralities of all nodes in the graph P_6 and $K_{2,3}$ (complete bipartite graph with one part with 2 nodes and the other part with 3 nodes).

P_6 :

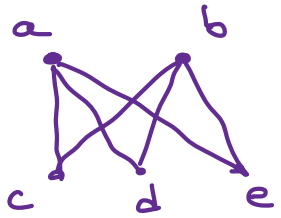
Degree: 1 2 2 2 2 1

Closeness: $\frac{1+2+3+4+5}{5}$, $\frac{1+1+2+3+4}{5}$, $\frac{1+1+2+2+3}{5}$,

$\frac{1+1+2+2+3}{5}$, $\frac{1+1+2+3+4}{5}$, $\frac{1+2+3+4+5}{5}$

Betweenness: $\frac{5}{\binom{6}{2}=15}$, $\frac{5+4}{15}$, $\frac{4+4+3}{15}$, $\frac{11}{15}$, $\frac{9}{15}$, $\frac{5}{15}$ (using symmetry)

$K_{2,3}$:

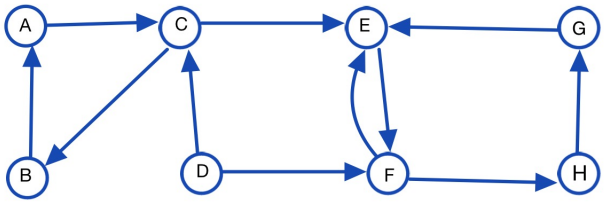


Degree:	$\frac{a}{3}$	$\frac{b}{3}$	$\frac{c}{2}$	$\frac{d}{2}$	$\frac{e}{2}$
Closeness:	$\frac{3\cdot 1+2}{5}=1$	1	$\frac{2\cdot 1+2\cdot 2}{5}=1.2$	1.2	1.2
Betweenness:	$\frac{4+\binom{3}{2}}{\binom{5}{2}}=\frac{7}{10}$	$\frac{7}{10}$	$\frac{4+1}{10}=\frac{5}{10}$	$\frac{5}{10}$	$\frac{5}{10}$

3. (a) (12 points) In the following directed graph, find all strongly connected components (SCC) using the algorithmic idea introduced in course notes.

   (b) (6 points) Merge the nodes in each SCC to a single supernode keeping the remaining edges as they are. Show that this new directed graph is acyclic (DAG) by obtaining a topological ordering of the vertices. (Note: A directed graph has a topological ordering if and only if it is a DAG.)

a)

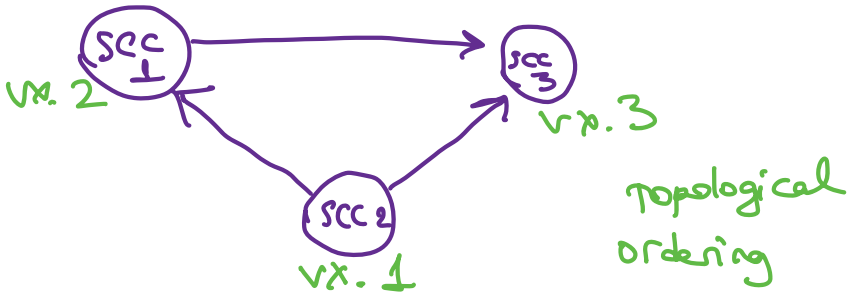


iter. 1:
vxs. reachable by A : {C, B, E, F, H, G, A}
vxs. reaching A : {B, C, D, A}
intersection: {C, B, A} SCC. 1

iter. 2:
vxs. reachable by D (from remaining): {E, F, H, G, D}
vxs. reaching D — // — : {D}
intersection : {D} SCC. 2

iter. 3:
vxs. reachable by E : {E, F, G, H}
vxs. reaching E : {E, F, G, H}
intersection : {E, F, G, H}
SCC. 3

b)

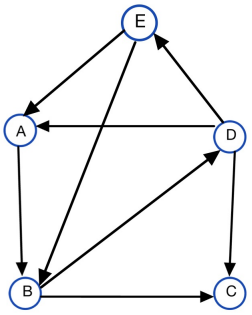


4. (20 points) Answer the following questions and show all your work.

   (a) Execute the first three iterations of the *power method* on the digraph (directed graph) below. (The first three iterations are already in the lecture slide for you to check your understanding.)

   (b) For $\epsilon = 0.1$, do you observe convergence after the first three iterations.

$$M = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1/3 & 1/2 \\ 1 & 0 & 0 & 0 & 1/2 \\ 0 & 1/2 & 0 & 1/3 & 0 \\ 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 0 \end{bmatrix} \end{array}$$



$$r_0 = \begin{bmatrix} 1/5 \\ 1/5 \\ 1/5 \\ 1/5 \\ 1/5 \end{bmatrix}$$

a) $r_1 = M \cdot r_0 = \frac{1}{5} \cdot \begin{bmatrix} (\frac{1}{3} + \frac{1}{2}) \\ (1 + \frac{1}{2}) \\ (\frac{1}{2} + \frac{1}{3}) \\ \frac{1}{2} \\ \frac{1}{3} \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 5/6 \\ 3/2 \\ 5/6 \\ 1/2 \\ 1/3 \end{bmatrix}$

$r_2 = M \cdot r_1 = \frac{1}{5} \begin{bmatrix} \frac{1}{3} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{3} \\ 1 \cdot \frac{5}{6} + \frac{1}{2} \cdot \frac{1}{3} \\ \frac{1}{2} \cdot \frac{3}{2} + \frac{1}{3} \cdot \frac{1}{2} \\ \frac{1}{2} \cdot \frac{3}{2} \\ \frac{1}{3} \cdot \frac{1}{2} \end{bmatrix} = \frac{1}{5} \begin{bmatrix} \frac{1}{3} \\ 1 \\ \frac{11}{12} \\ \frac{3}{4} \\ \frac{1}{6} \end{bmatrix}$

$r_3 = M \cdot r_2 = \frac{1}{5} \begin{bmatrix} \frac{1}{3} \cdot \frac{3}{4} + \frac{1}{2} \cdot \frac{1}{6} \\ \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{6} \\ \frac{1}{2} \cdot 1 + \frac{1}{3} \cdot \frac{3}{4} \\ \frac{1}{2} \cdot 1 \\ \frac{1}{3} \cdot \frac{3}{4} \end{bmatrix} = \frac{1}{5} \begin{bmatrix} \frac{1}{3} \\ \frac{5}{12} \\ 3/4 \\ 1/2 \\ 1/4 \end{bmatrix}$

$(0 + 7/12 + 2/12 + 3/12 + 1/12) = \frac{13}{12}$

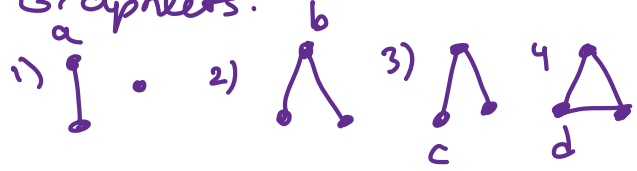
b) $|r_3 - r_2| = \frac{1}{5} \left[\left|\frac{1}{3} - \frac{1}{3}\right| + \left|1 - \frac{5}{12}\right| + \left|\frac{11-9}{12}\right| + \left|\frac{3-2}{4}\right| + \left|\frac{2-3}{12}\right| \right]$

NO CONVERGENCE YET

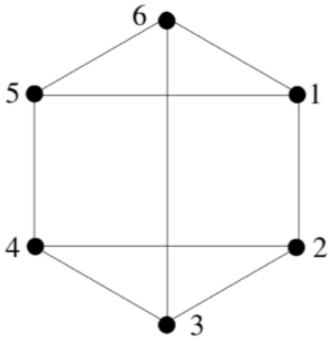
5. (a) (10 points) Find the GDV (Graphlet Degree Vector) for each vertex in the sample graph for all graphlets with 3 nodes. (You can make use of symmetry.)

   (b) (12 points) Calculate $cut(A, B)$, $vol(A)$, $vol(B)$, and $\phi(A, B)$ (conductance) for the graph below, with 1) $A = \{1, 2, 3\}$ and $B = \{4, 5, 6\}$ and 2) $A = \{1, 5, 6\}$ and $B = \{2, 3, 4\}$.

a) Graphlets:



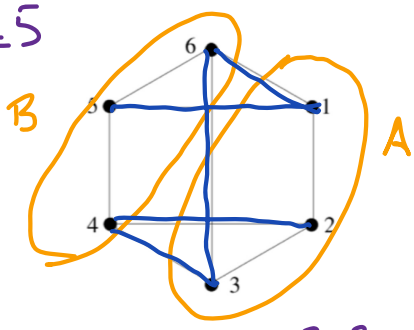
	1	2/3	3	4/3	5/3	6/2
a)	2+1=3		1+1=2			
b)	2	2	2	2	2	2
c)	1+1+2=4	4	1+2+1=4	4	4	4
d)	1	1	1	1	1	1



b) 1) $vol(A) = 3+3+3 = 9$, $vol(B) = 3+3+3 = 9$

$cut(A,B) = 5$

$\phi(A,B) = \frac{5}{9}$



2) $vol(A) = 3 \cdot 3 = 9$, $vol(B) = 3 \cdot 3 = 9$

$cut(A,B) = 3$

$\phi(A,B) = \frac{3}{9}$