BBM401-Lecture 3: Nondeterminism: NFA definitions, e-transitions, equivalence with DFAs

Lecturer: Lale Özkahya

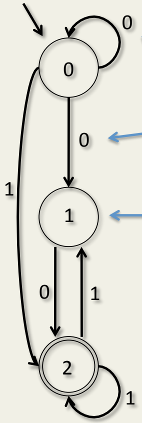
Resources for the presentation:
http://ocw.mit.edu/courses/electrical-engineering-and-computer-science/6-045j-automata-computability-and-complexity-spring-2011/Syllabus/
https://courses.engr.illinois.edu/cs498374/lectures.html

DFA Reminder

A DFA is a quintuple $M=(Q,\Sigma,\delta,q_0,F)$, *where:*

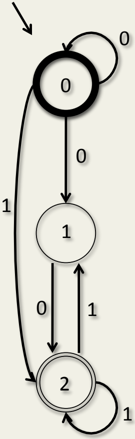
- Q is a finite set of *states*
- Σ is a finite *alphabet* of symbols
- $\delta: Q \times \Sigma \rightarrow Q$ is a *transition function*
- q_0 is the *initial state*
- $F \subseteq Q$ is the set of *accepting states*

NFA: NONDETERMINISTIC FINITE AUTOMATON



state 0 has two possible actions on input 0

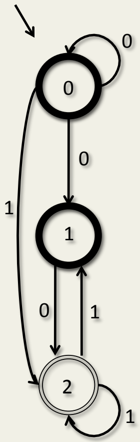
state 1 has NO possible actions on input 1



Let's see how a computation proceeds

What is the next state???

INPUT $\boxed{0}00110$



Two views:

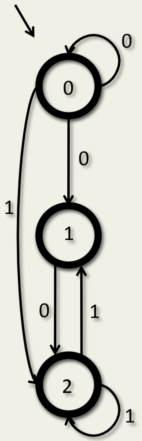
(a) Possible worlds

EITHER *could be* the next state.

(b) Parallel threads

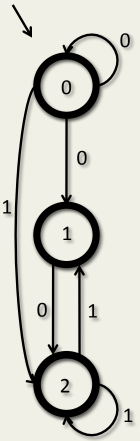
BOTH *are* “the next state”; the NFA spawns a second thread – it is in two states at the same time.

INPUT 000110

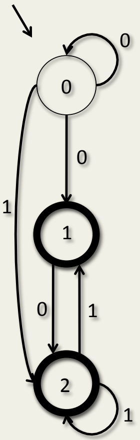


After reading 00, the machine could be in ANY of its states!

INPUT 000110

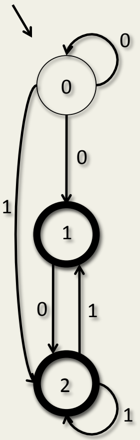


INPUT 000$\boxed{1}$10

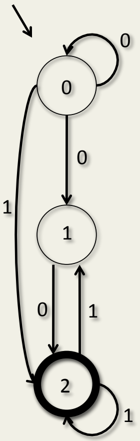


One of the threads has died

INPUT 000110



INPUT 000110



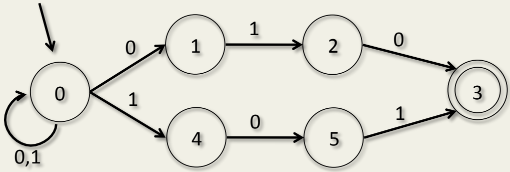
INPUT 000110

Formalism

- NFA is same as DFA, except transition δ returns a *set of possible next states:*
- $\delta: Q \times \Sigma \to 2^{Q}$ so that $\delta(q,a) \subseteq Q$
- We write $q \xrightarrow{a} p$ if p *is in* $\delta(q,a)$
- Everything else is unchanged. But now for a string *w,* there may be many states p such that
  $q \overset{w}{\leadsto} p$. (Exists a path labeled w leading from q to p)

N accepts w if *for some* f in F, $q_0 \overset{w}{\leadsto} f$.

Example NFA N



What strings can *possibly* end at state 3?

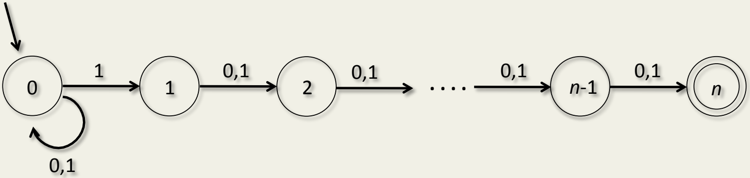
$L(N) = \{w \mid w$ ends with "010" or with "101"$\}$

must make sure:

(1) *every* string with either ending *can* be accepted.

(2) *every* string without either ending *cannot* possibly be accepted.

Example NFA N



What strings can *possibly* end at state n?

$\boxed{L(N) = \{w : w\text{'s } n^{th} \text{ from last character is a "1"}\}}$

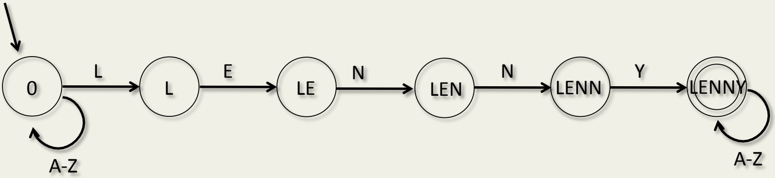
requires 2^n DFA states

must make sure:

(1) *every* string with 1 in n^{th}-from-last position *can* be accepted.
(2) *every* string with a 0 in n^{th}-from-last position *cannot* possibly be accepted.

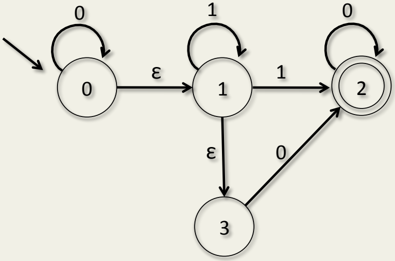
Challenge NFA Construction

Create an NFA that recognizes the set of strings that contain your FIRST NAME.



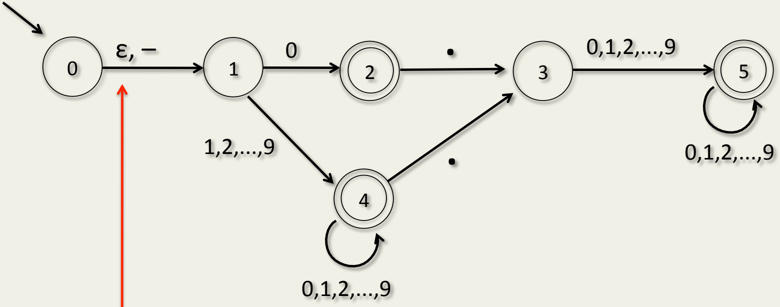
ε-NFAs: NFAs with ε-edges

- Allow transition without reading character
- $\delta: Q \times (\Sigma \cup \{\varepsilon\}) \rightarrow 2^Q$



On empty input, which states can be reached?
On input 0, which states can be reached?

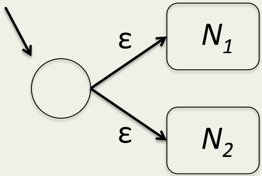
Accept decimal numbers



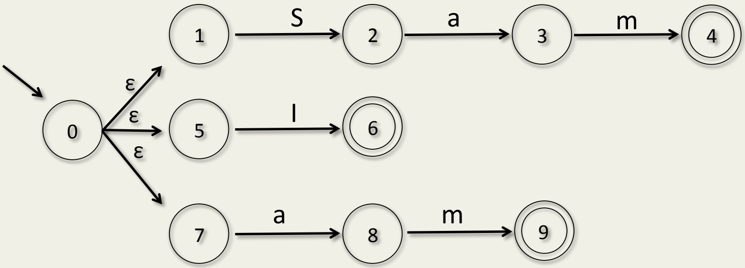
useful when an input symbol is optional

Utility of ε-edges

- nondeterministically choose between two cases
- construct NFA for case 1, NFA for case 2, then add new start state with ε-edges to choose which NFA to "run".
- accepts the union of the languages

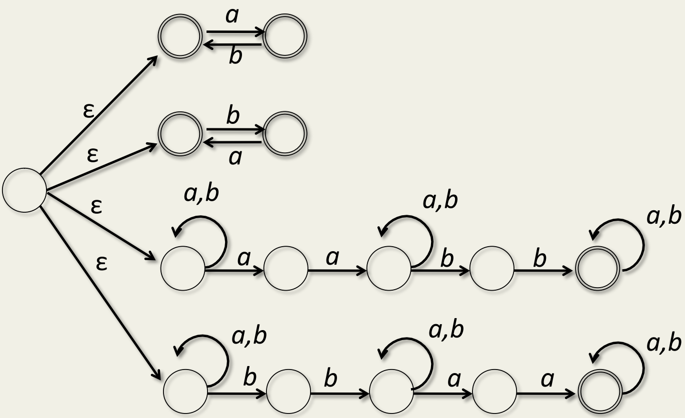


Example: accept one of several keywords

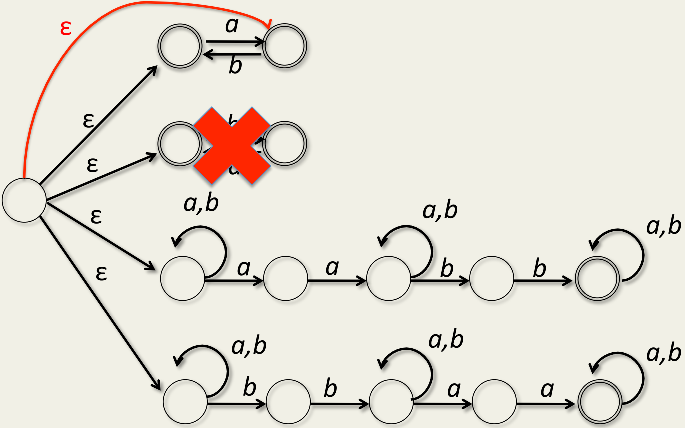


No need to think about how different keywords overlap.
Can essentially have several starting states.

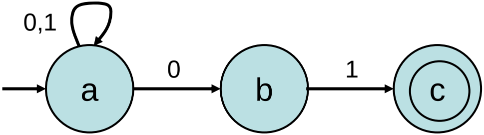
L(N) = {w | either contains both *aa* and *bb*, or neither}



L(N) = {w | contains either both *aa* and *bb*, or neither}



NFA Example 1



$Q = \{ a, b, c \}$

$\Sigma = \{ 0, 1 \}$

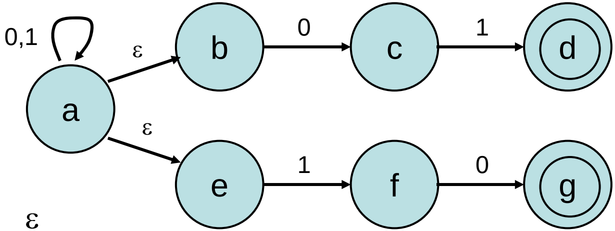
$q_0 = a$

$F = \{ c \}$

δ:

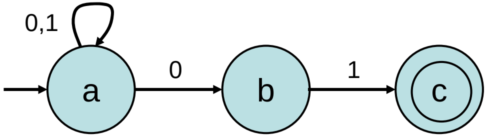
	0	1	ε
a	{a,b}	{a}	∅
b	∅	{c}	∅
c	∅	∅	∅

NFA Example 2



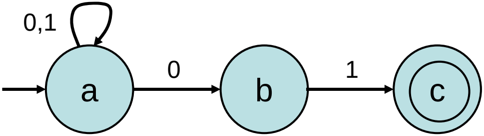
	0	1	ε
a	{a}	{a}	{b,c}
b	{c}	∅	∅
c	∅	{d}	∅
d	∅	∅	∅
e	∅	{f}	∅
f	{g}	∅	∅
g	∅	∅	∅

Example 1



- L(M) = { w | w ends with 01 }
- M accepts exactly the strings in this set.
- Computations for input word w = 101:
  - Input word w: 1 0 1
  - States: a a a a
  - Or: a a b c
- Since c is an accepting state, M accepts 101

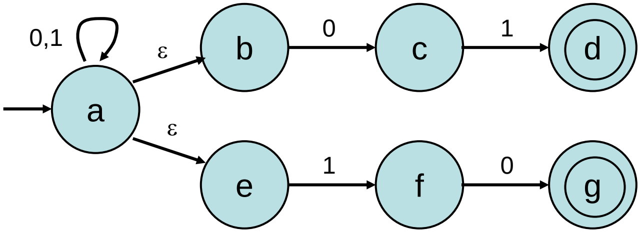
Example 1



- Computations for input word w = 0010:
  - Possible states after 0: { a, b }
  - Then after another 0: { a, b }
  - After 1: { a, c }
  - After final 0: { a, b }
- Since neither a nor b is accepting, M does not accept 0010.

$$\{ a \} \xrightarrow{0} \{ a, b \} \xrightarrow{0} \{ a, b \} \xrightarrow{1} \{ a, c \} \xrightarrow{0} \{ a, b \}$$

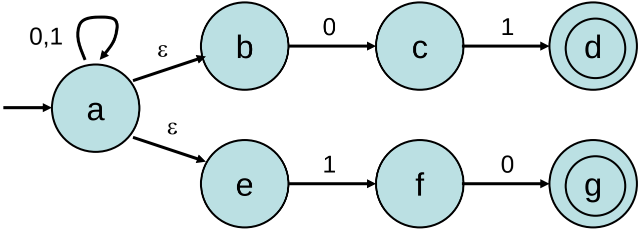
Example 2



- L(M) = { w | w ends with 01 or 10 }
- Computations for w = 0010:
  - Possible states after no input: { a, b, e }
  - After 0: { a, b, e, c }
  - After 0: { a, b, e, c }
  - After 1: { a, b, e, d, f }
  - After 0: { a, b, e, c, g }
- Since g is accepting, M accepts 0010.

$$\{ a, b, e \} \xrightarrow{0} \{ a, b, e, c \} \xrightarrow{0} \{ a, b, e, c\} \xrightarrow{1} \{ a, b, e, d, f \} \xrightarrow{0} \{ a, b, e, c, g \}$$

Example 2



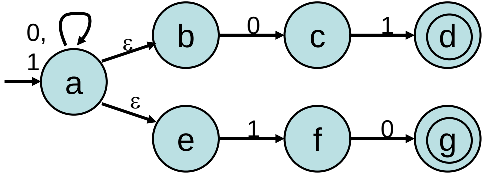
- Computations for w = 0010:

$$\{ a, b, e \} \xrightarrow{0} \{ a, b, e, c \} \xrightarrow{0} \{ a, b, e, c \}$$
$$\xrightarrow{1} \{ a, b, e, d, f \} \xrightarrow{0} \{ a, b, e, c, g \}$$

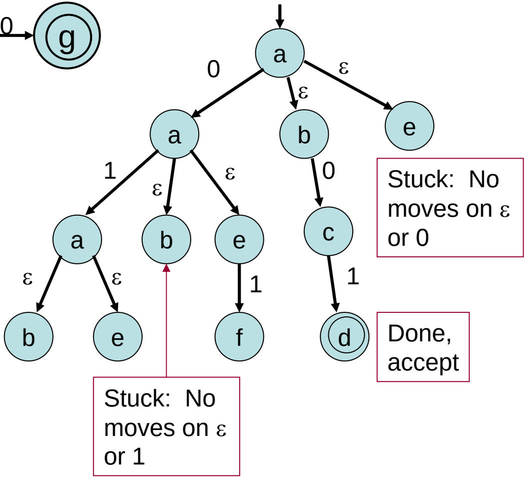
- Path to accepting state:

$$a \xrightarrow{0} a \xrightarrow{0} a \xrightarrow{\varepsilon} e \xrightarrow{1} f \xrightarrow{0} g$$

Viewing computations as a tree



Input w = 01



In general, accept if there is a path labeled by the entire input string, possibly interspersed with εs, leading to an accepting state.

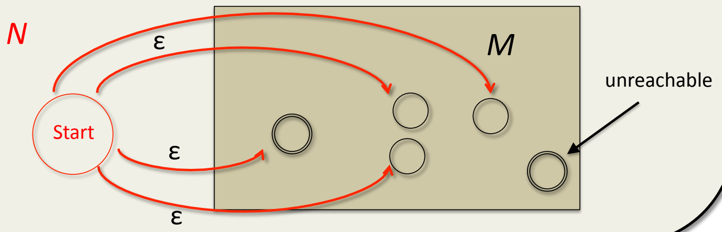
Here, leads to accepting state d.

Closure under Suffixes

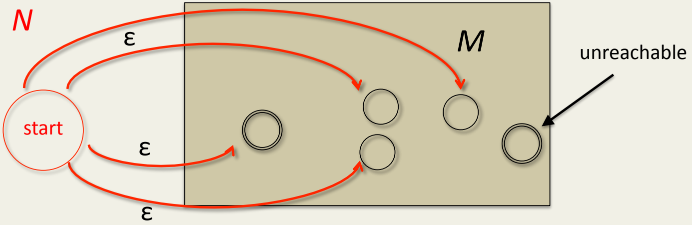
- Show that if L is accepted by some DFA M, then there is an NFA N that accepts

  suffixes(L) = {w | w is a suffix of some word in L}

- Proof by constructing N from M.



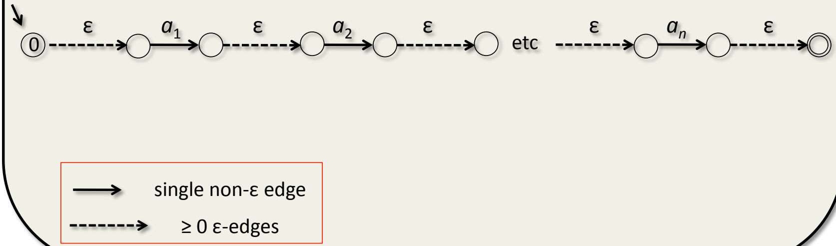
Closure under Suffixes



- Let $M = (Q, \Sigma, \delta, q_{0,} F)$, then $N = (Q^N, \Sigma^N, \delta^N, q^N_{0,}, F^N)$
  - $Q^N =$ Q U {start}
  - $\Sigma^N = \Sigma$
  - $\delta^N =$ $\delta(q,a)$ for q in Q; $\delta^N(start, \varepsilon)$ = {q | q reachable}
  - $q^N_0 =$ start
  - $F_N = F$

Informal Definition of acceptance

N an ε-NFA accepts a string $w=a_1a_2...a_n$ iff there is a path (perhaps including ε-edges) from state 0 to an accepting state, such that the concatenation of symbols along the path = $a_1a_2...a_n$



ε-edges do not increase power

Theorem

If L is recognized by some NFA N with ε-edges, then there is an equivalent NFA N′ without ε-edges that recognizes L

Proof is by SIMULATION of *N* by an *N′*

Construction

Given ε-NFA $N = (Q,\Sigma,\delta,q_0,F)$
Construct NFA $N' = (Q,\Sigma,\delta',q_0,F')$:

$F' = F$ unless $0 \xrightarrow{\varepsilon} \circledcirc$
in which case $F' = F \cup \{q_0\}$

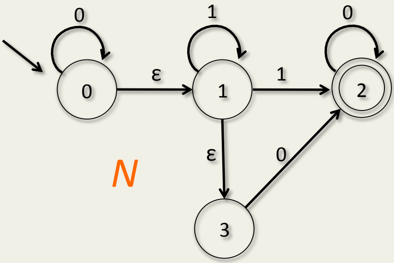
δ' extends δ by adding transitions to compensate for lack of ε-edges.

δ' contains all non-ε-edges of δ, but in addition:
for every p,q in Q, for every a in Σ, if there is a path

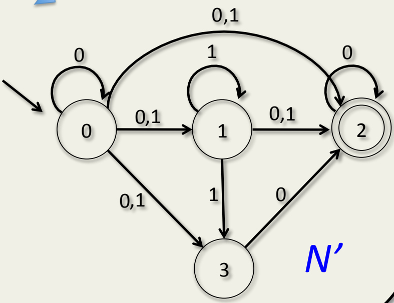
$$p \xrightarrow{\varepsilon} \bigcirc \xrightarrow{a} \bigcirc \xrightarrow{\varepsilon} q \quad \text{in } N$$

then add $p \xrightarrow{a} q$ to N'

Example: eliminating ε-edges



copy over, omitting ε-edges



state 0 on input 0 can reach states 0,1,2,3
state 0 on input 1 can reach states 1,2,3
state 1 on input 0 can reach state 2
state 1 on input 1 can reach states 1,2,3
state 2 on input 0 can reach state 2
state 2 on input 1 can reach nothing
state 3 on input 0 can reach state 2
state 3 on input 1 can reach nothing

and NOT $0 \overset{\varepsilon}{\dashrightarrow} \circledcirc$ so $F' = F$

Proof that simulation works

- Prove that L(N') = L(N)
- Use the following

Lemma

For all states p,q, and for all NONEMPTY strings w

$p \overset{w}{\dashrightarrow} q$ in N if and only if $p \xrightarrow{w} q$ in N'

perhaps ε-edges

no ε-edges

DEF for M:

$p \xrightarrow{a} q$ in N' iff

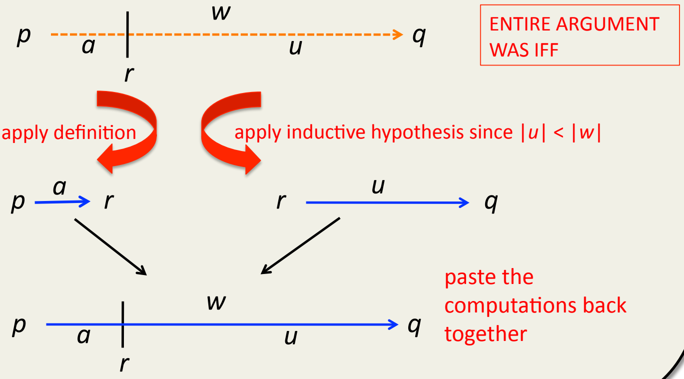
$p \overset{a}{\dashrightarrow} q$ in N

BY INDUCTION:

To show

$p \xrightarrow{w} q$ in N' iff

$p \overset{w}{\dashrightarrow} q$ in N



Since lemma is true....

Show for every w: N' accepts w if and only if N accepts w

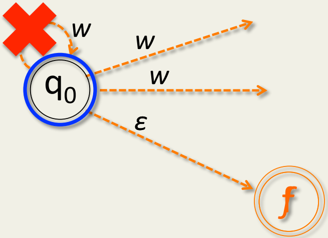
If NOT $q_0 \xrightarrow{\varepsilon} f$ in F then $F' = F$ and:

- Neither N nor N' accept ε
- If $|w| > 0$. By Lemma, w goes to same states in N' as it did in N, and since $F' = F$, it is accepted by N' iff it was accepted by N

Since lemma is true....

If $q_0 \xrightarrow{\varepsilon} f$ in F then $F' = F \cup \{q_0\}$

- Case 1: $w=\varepsilon$, so is accepted by both N and N'
- Case 2: $|w| > 0$, then
  - Case 2a: NOT $q_0 \xrightarrow{w} q_0$ in N.

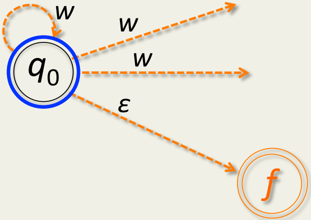


By Lemma, w doesn't reach q_0 in N' either
So adding q_0 to F' didn't change w's acceptance

Since lemma is true....

If $q_0 \xrightarrow{\varepsilon} f$ in F then $F' = F \cup \{q_0\}$

- Case 1: $w=\varepsilon$, so is accepted by both N and N'
- Case 2: $|w| > 0$, then
  - Case 2b: $q_0 \xrightarrow{w} q_0$ in N.



In N', q_0 was made accepting
But $q_0 \xrightarrow{\varepsilon} f$ in F anyway
So w is accepted by both

Eliminating Nondeterminism

NFA → DFA Theorem

Theorem

If L is recognized by some NFA N, then there is an equivalent DFA that recognizes L !!

Nondeterminism doesn't increase the computational power of finite automata

Proof is by SIMULATION of an NFA by a DFA

assume wlog that NFA has no ε-edges

But first....

How would you write a program to tell if a word *w* was accepted by some DFA *M* ?

How is DFA represented?

DFA is table δ (2d array) $Q \times \Sigma$ with constant access time to get $\delta(q,a)$

F is boolean array indicating accept

state \ input	0	1
0	1	0
1	2	0
2	3	0
3	3	0

But first....

How would you write a program to tell if a word w was accepted by some DFA M ?

input / state	0	1
0	1	0
1	2	0
2	3	0
3	3	0

Alg M (*delta*: array; w: string)

state = 0

for i = 1 to $|w|$

state = *delta*(state,w_i)

output F(state)

How about NFAs?

How would you write a program to tell if a word *w* was accepted by some **NFA** *N*?

- What is representation of an NFA?
- NFA has table δ (2d array) Q x Σ where each entry is a pointer to a linked list of possible next states $\delta(q,a)$
- Time to collect "next states" from q, a is $O(n)$

Algorithms for NFA computing

ACCEPTS (N,w)

  Return ACCEPT?(q_0,w)

ACCEPT? (q,w)   /* does w go to an accepting state from q */

  if $w = \varepsilon$

    return $F(q)$

  else $w = au$,

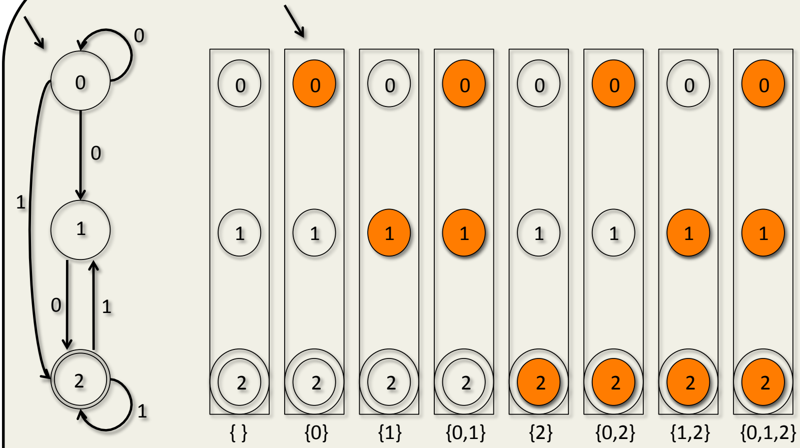
    return OR { ACCEPT? (p, u) | p in $\delta(q,a)$ }

- Proof that this is correct?   simple induction on $|w|$
- Time taken by this algorithm?   don't ask; don't tell (until later)

Algorithms for NFA computing

```
ACCEPTS (N,w)
    active, new_active = sets of states implemented as
                         boolean arrays indexed by states
    active = [1,0,0,0,0,...,0]  (initially, only starting state 0 is active)
    for i = 1 to |w|
        new_active = Ø  (zero out the array)
        for each state q in active
                put each element of δ(q,w_i) in new_active
        active = new_active
    if active contains an accepting state, then return TRUE
                                           else return FALSE
```

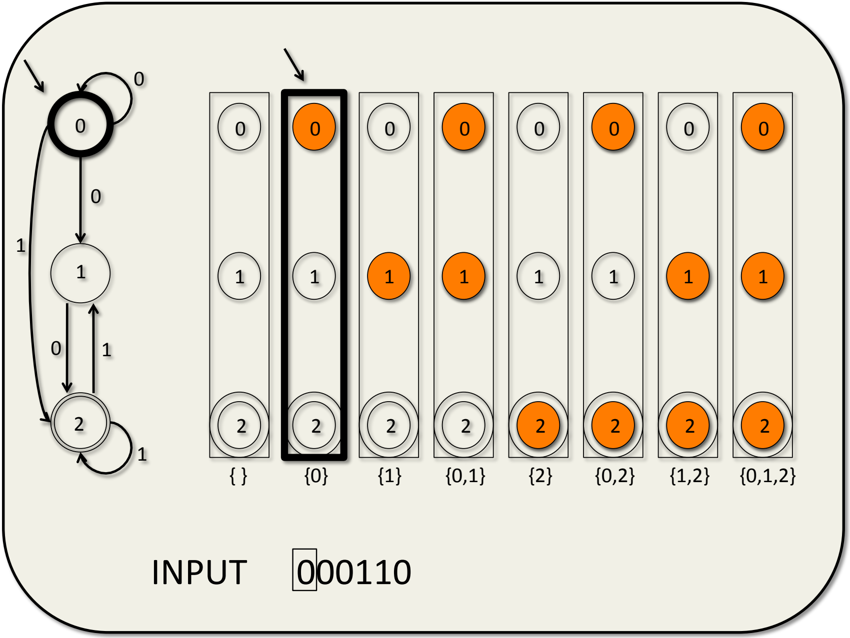
- Proof that this is correct? simple induction on $|w|$
- Time taken by this algorithm? $O(n^2 w)$



NFA

STATES OF THE EQUIVALENT “POWER-SET” DFA

A state corresponds to a subset of states of the NFA, showing which are active threads (a boolean array)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{0}
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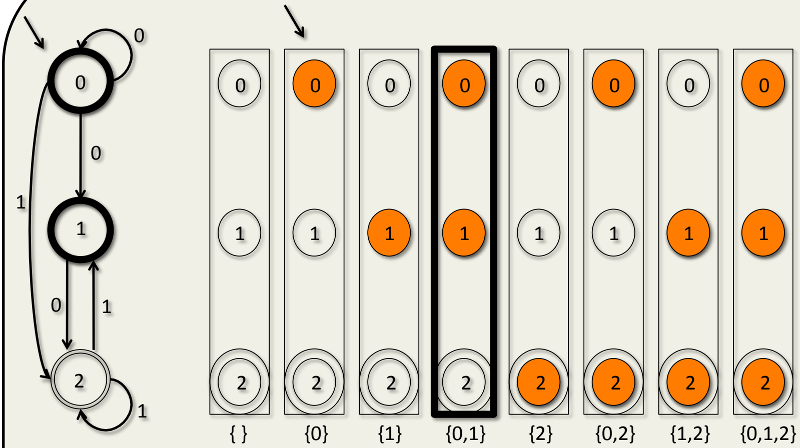
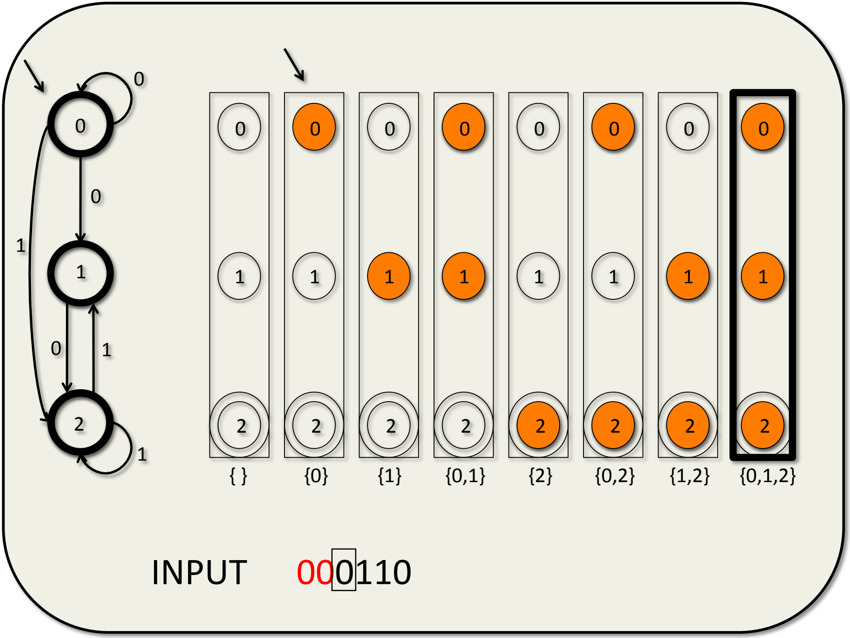
{2}
{0,2}
{1,2}
{0,1,2}
INPUT 000110



INPUT 000110



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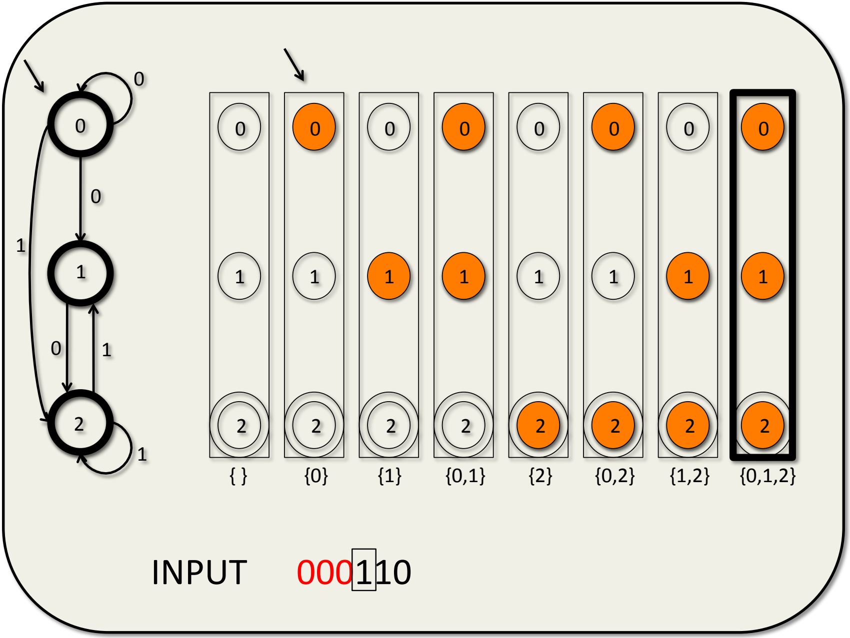
{1,2}
{0,1,2}
INPUT
000110



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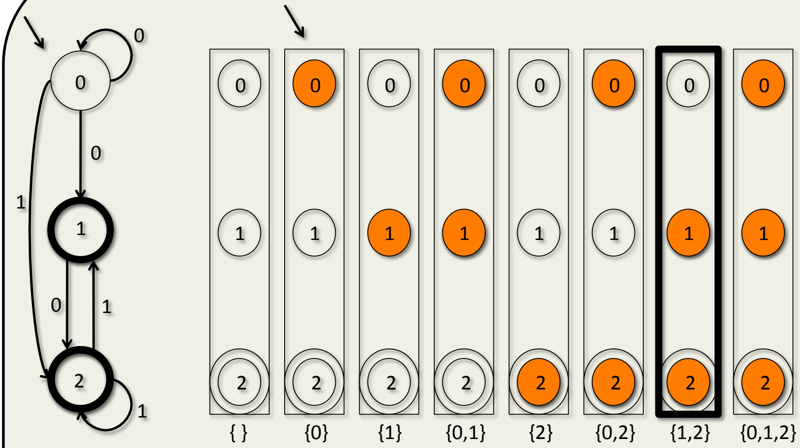
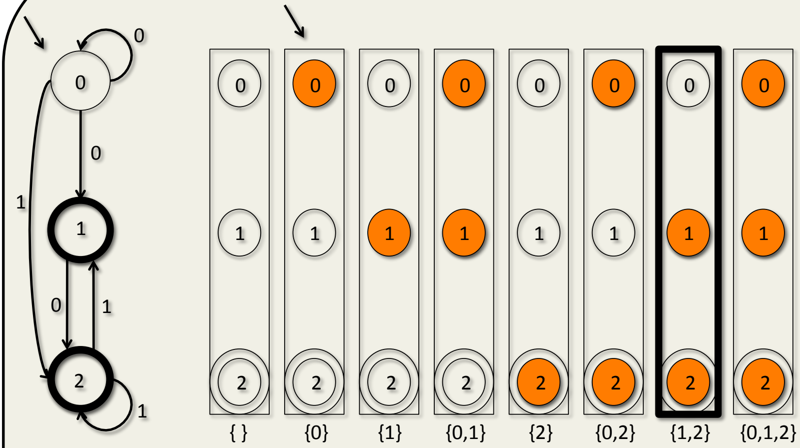
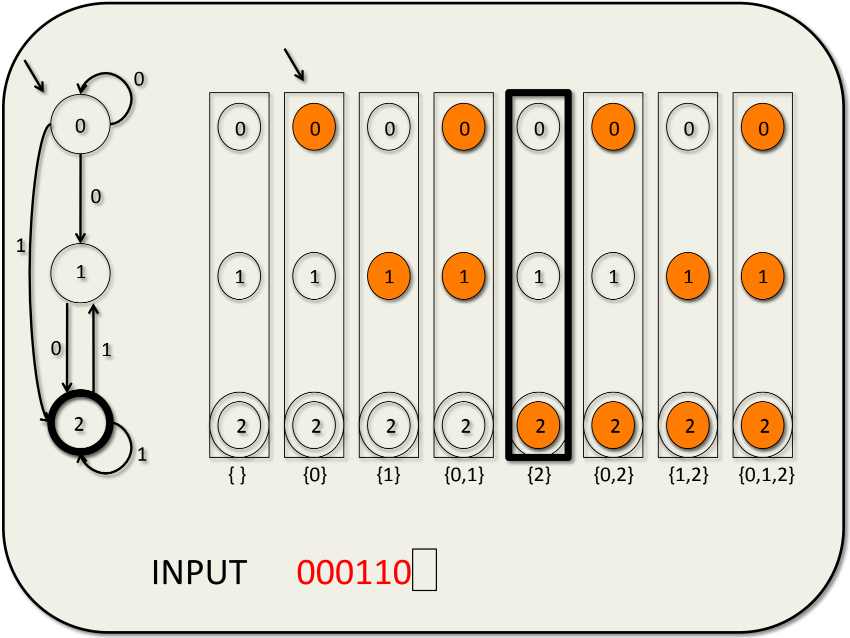
{1,2}
{0,1,2}
INPUT 000110



INPUT 000110



INPUT 000110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0 1 2
{1,2}
0 1 2
{0,1,2}
INPUT
000110

Formal specification

- Let $N = (Q,\Sigma,\delta,q_0,F)$ be an NFA
- Recall that δ: $Q \times \Sigma \rightarrow 2^Q$, so that

  $\delta(q,a)$ is a set of possible states.

- Build $M = (Q',\Sigma,\delta',q'_0,F')$ that simulates N:
  - $Q' = 2^Q$ (the power set of Q)
  - Σ is the same
  - $q'_0 = \{q_0\}$
  - $F' = \{S \subseteq Q\text{: } S \cap F \neq \emptyset\}$

if *N could have* ended in an accepting state, then *S* will contain an element of *F*

since *S* is in *Q'*, it is a subset of *Q*

Formal specification

- Let $N = (Q,\Sigma,\delta,q_0,F)$ be an NFA
- $M = (Q',\Sigma,\delta',q'_0,F')$ What is δ' ??

$\delta'(S,a) =$

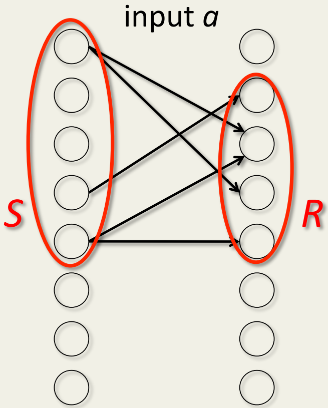
for every state s in S

all the states to which s can go on input a

$S \xrightarrow{a} R$ iff $R = \{ r\text{: } s \xrightarrow{a} r$ for some s in $S\}$

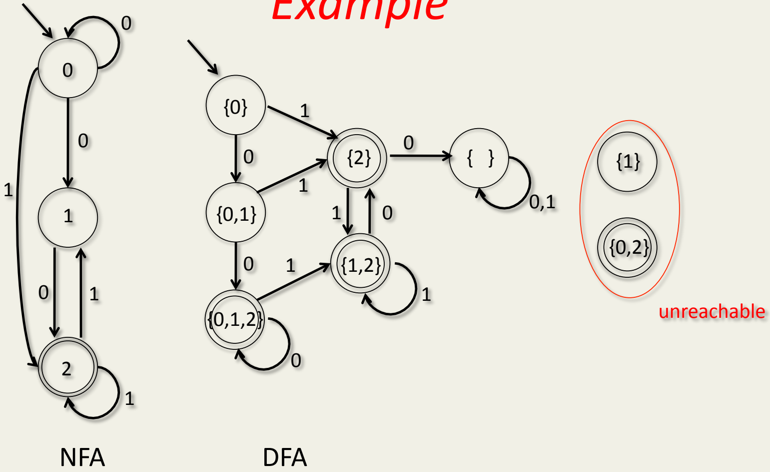
$S \xrightarrow{a}$ *exactly the set of states that N can reach from anything in S on input a*

Intuition



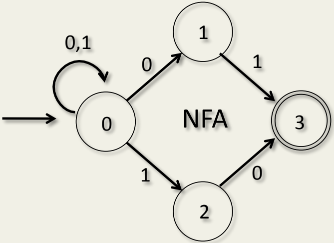
Then set $S \xrightarrow{a} R$ in M

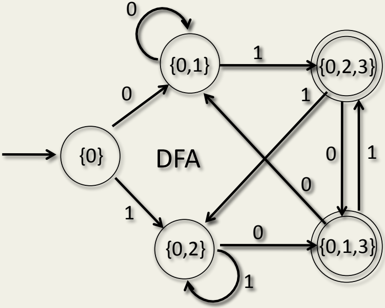
Example



Which are the accepting states?

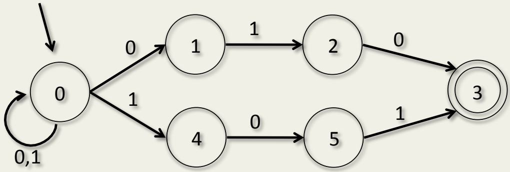
Challenge





Which are the accepting states?

Challenge NFA
(do at home)



Be careful – it is easy to get confused

To show

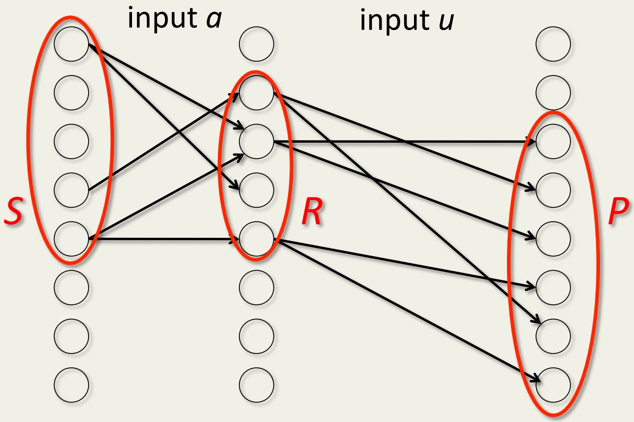
- "Action" of DFA is correct
- What would that mean?
- $\{q_0\} \overset{w}{\leadsto} P$ = {states NFA could be in}
- $= \{p : \ q_0 \overset{w}{\leadsto} p\}$
- But more generally, for *any* state S of DFA

  (set of states of NFA)

  $S \overset{w}{\leadsto} P = \{p$: for some s in S, $s \overset{w}{\leadsto} p\}$

$S \overset{w}{\leadsto}$ **exactly** the states that N could reach starting at a state in S

Intuition (w = au)



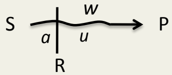
DEF for M:

$S \xrightarrow{a}$ *exactly the states that N can reach from S on input a*

BY INDUCTION:

To show

$S \overset{w}{\leadsto}$ *exactly the states that N can reach from S on input w*



pull apart the computation

$S \xrightarrow{a} R$

apply definition

R = the reachable states from S on a

$R \overset{u}{\leadsto} P$

apply inductive hypothesis since $|u| < |w|$

P = the reachable states from R on u

p in P iff p can be reached via u from some state r in R.
r in R iff r can be reached via a from some state s in S

paste the computations back together

SO, p in P iff p can be reached via au from some s in S

Finishing up....

$S \overset{w}{\rightsquigarrow}$ *exactly the set of states that N can reach from S on input w*

$\{q_0\} \overset{w}{\rightsquigarrow}$ *exactly the set of states that N can reach from* q_0 *on input w*

N accepts a string *w*

iff in N, $q_0 \overset{w}{\rightsquigarrow} f$ for some f in F

iff in M, $\{q_0\} \overset{w}{\rightsquigarrow} S$ for some S containing f

iff in M, $\{q_0\} \overset{w}{\rightsquigarrow} S$ in F' (recall $F' = \{S \subseteq Q : S \cap F \neq \emptyset\}$)

iff *M* accepts *w*